The Physics of Higher-Spin Theories

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von

Pan Kessel

Präsidentin der Humboldt-Universität zu Berlin:
Prof. Dr. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät:
Prof. Dr. Elmar Kulke

Gutachter/innen: 1. Prof. Dr. Hermann Nicolai

2. Prof. Dr. Jan Plefka

3. Dr. Xavier Bekaert

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THE PHYSICS OF HIGHER-SPIN THEORIES

PAN KESSEL

Pan Kessel: *The Physics of Higher-Spin Theories*

ABSTRACT

Higher-spin theories have received significant attention over the last years. This is because they arise as the bulk duals of comparatively tractable conformal field theories.

The only known interacting higher-spin theories were constructed by Vasiliev and are formulated in a highly non-standard way in terms of an infinite number of auxiliary fields.

This thesis extracts physics out of Vasiliev theory. We study in detail its interactions, spectrum and locality properties. We consider both the three- and four-dimensional case. Our work represents the first systematic study of Vasiliev theory at the interacting level (in terms of physical fields only).

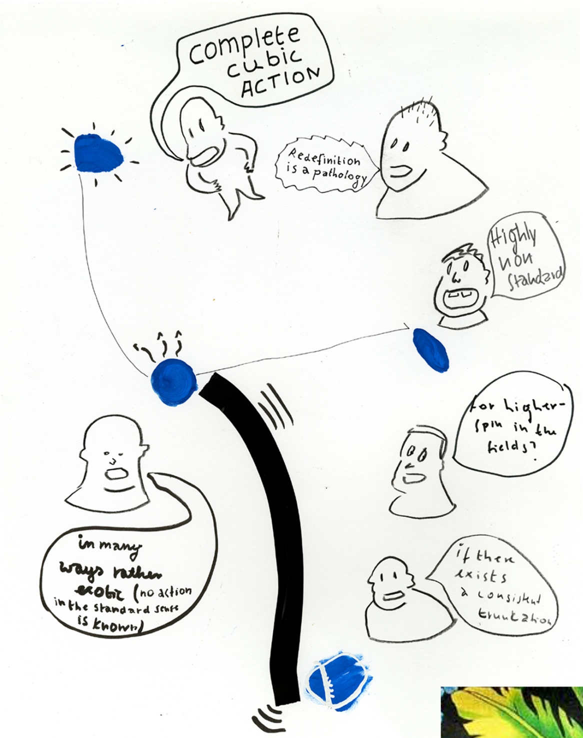
ZUSAMMENFASSUNG

Höhere Spin Theorien haben in den letzten Jahren große Aufmerksamkeit gefunden. Ein Grund dafür ist, dass diese Theorien dual zu besonders einfachen konformen Feldtheorien sind.

Die einzigen bekannten wechselwirkenden höheren Spin Theorien wurden von Vasiliev in einem sehr ungewöhnlichen Formalismus und mit unendlich vielen Hilfsfeldern konstruiert.

Die vorliegende Arbeit extrahiert die Physik, die durch diese Gleichungen beschrieben wird. Wir untersuchen im Detail die Wechselwirkungen, das Spektrum sowie die Lokalitätseigenschaften der Vasiliev Theorie. Diese Arbeit ist die erste systematische Untersuchung der Vasiliev Theorie auf wechselwirkender Ebene (nur ausgedrückt durch physikalische Felder).

I sent out the technical parts of introduction to this thesis to my friends with an invitation to come up with a "piece of art" which to their mind summarizes what I have been up to over the last years. Here are some of the results:



1 • Simon
2 • Bene
3 • Karin

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1 INTRODUCTION

1.1 MOTIVATIONS

In this thesis, we will study classical field theories with massless spin-s gauge fields in both three and four dimensions. In particular, we will consider higher-spin gauge fields, i.e. massless fields of spin greater than two. These fields can be described by completely symmetric fields $\phi_{n_1 \dots n_s}$ with the spin-s gauge transformations

$$\delta\phi_{n_1 \dots n_s} = \nabla_{(n_1}\xi_{n_2 \dots n_s)} \, . \tag{1.1}$$

Field theories containing higher-spin gauge fields are of great interest for at least three (not completely unrelated) reasons:

- They provide a natural generalization of gauge theory which underlies all fundamental interactions in nature.
- They might lead to a better understanding of string theory.
- They arise in particularly simple examples of gauge/gravity dualities.

Let us discuss these reasons in more detail. Our current understanding of the fundamental processes underlying all phenomena in nature is based on two highly successful theories: the standard model of particle physics and general relativity. The standard model describes all known interactions but gravity and is formulated in terms of Yang-Mills gauge theories (coupled to a certain matter sector). One of the most impressive achievements of modern experimental particle physics has been to verify the standard model to a remarkable level of precision. In the standard model, the exchange of spin-1 gauge fields leads to the strong and electroweak force. Similarly, gravitational interactions can perturbatively be described by the exchange of spin-2 gauge bosons, the gravitons, and the corresponding gauge theory is general relativity. Since Einstein's original formulation more than 100 years ago, general relativity has been successfully tested in a large number of experiments - most spectacularly by the recent detection of gravitational waves by the LIGO collaboration. Given the pivotal roles of spin-1 and spin-2 gauge theories in nature, it is tempting to study field theories with spin greater than two. For a long time, it was thought that interacting higher-spin gauge theories cannot be constructed. This belief was based on a large number of highly restrictive no-go results (see [1] for a modern review). However, over the last decades explicit examples of such theories have been constructed by Vasiliev [2, 3] and therefore provide us with an additional class of interacting classical gauge theories. It is an open question whether all interacting higher-spin theories can be described by Vasiliev-like theories. Currently, no counter-example is known.

String theory contains an infinite tower of massive higher-spin fields with masses $M^2 \sim l_s^{-2}$, where l_s is the string length. Typically, one considers the point particle limit in which l_s is taken to be small compared to the length scale we are interested in. In this regime, the higher-spin fields become very massive and are therefore irrelevant for low-energy physics. However, there is also the opposite limit of l_s much greater than the physical length scale. In this tensionless limit, all higher-spin fields of string theory are massless and the theory therefore possesses a huge higher-spin gauge symmetry. It is widely believed that this is the underlying gauge algebra of string theory and by higgsing this gauge symmetry the infinite tower of higher-spin fields becomes massive. Over the last years, this Higgs mechanism has become a very active and exciting field of research [4–6]. This was achieved by comparing the dual conformal field theories of particular higher-spin and string theories (in the tensionless limit and on certain backgrounds). In these conformal field theories, the questions discussed above can be analyzed quantitatively.

The dualities between higher-spin theories in the bulk and conformal field theories on the boundary can be motivated by the following reasoning [7]: we start from the celebrated Maldacena duality of four-dimensional $\mathcal{N} = 4$ Super-Yang-Mills with gauge group $SU(N)$ and type IIB string theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. $\mathcal{N} = 4$ Super-Yang-Mills is a (super-)conformal theory with gauge coupling g_{YM}. The parameters of type IIB string theory, the string coupling g_s and string length l_s, are then related to the free parameters of the gauge theory by

$$g_{\mathrm{YM}}^2 = 2\pi g_s \,, \qquad\qquad 2 g_{\mathrm{YM}}^2 N = \left(\frac{l}{l_s}\right)^4 \,, \tag{1.2}$$

where l denotes the AdS radius. String theory is currently best understood in the regime of small g_s which corresponds to small g_{YM} of the gauge theory. Mostly, one then considers the point particle limit $l_s/l \to 0$. This corresponds to the limit of strong 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ in the gauge theory. Therefore, the point pariticle limit of string theory at weak coupling is related to the gauge theory at strong coupling. While this feature is appealing as it provides us with a tool to study strongly coupled gauge theory, it also poses a significant challenge in checking the duality. On the other hand, one may also try to consider the tensionless limit $l_s/l \to \infty$. In this limit, the 't Hooft coupling λ vanishes and the Yang-Mills theory becomes free. But, as we have discussed before, the corresponding string theory now contains an infinite number of massless higher-spin fields. Presently, the higher-spin theory describing type IIB string theory on this background is not known, but this reasoning led Klebanov and Polyakov to propose a duality between a certain higher-spin theory (type A minimal Vasiliev theory) on AdS_4 and the singlet sector of the three-dimensional free $O(N)$ vector model (the theory of N free bosons transforming in the fundamental representation of $O(N)$) in the large N limit [8]. By changing boundary conditions of the bulk scalar field, the Vasiliev theory is no longer dual to the free but the critical vector model. In spirit, these dualities are very similar to the tensionless limit of the Maldacena duality as they relate weakly coupled conformal field theories to weakly coupled higher spin theories. By now, many generalizations of this duality have been found: Sezgin and Sundell proposed that another Vasiliev theory (type B Vasiliev theory) is dual to a fermionic version of the vector model [9]. Supersymmetric extensions of these dualities as well as correspondences of parity violating versions of Vasiliev theory to vector models gauged by a Chern–Simons theory have also been studied [4, 10, 11].

Inspired by the Klebanov–Polyakov conjecture, Gaberdiel and Gopakumar proposed a duality between two-dimensional $\mathcal{W}_N$-minimal models and three-dimensional Vasiliev theories with gauge algebra $\mathfrak{hs}(\lambda)$ [12–14].[1] More precisely, $\mathcal{W}_N$-minimal models are two-dimensional conformal field theories which are given by Wess–Zumino–Witten coset models of the form

$$\frac{\mathrm{SU}(N)_k \otimes \mathrm{SU}(N)_1}{\mathrm{SU}(N)_{k+1}} \,. \tag{1.3}$$

This conjecture was put forward for the 't Hooft limit of these theories in which $N, k \to \infty$ at fixed λ, where

$$0 \leq \lambda = \frac{N}{N+k} \leq 1 \,. \tag{1.4}$$

The above 't Hooft coupling is to be identified with the λ parameter of the $\mathfrak{hs}(\lambda)$ Vasiliev theory. Various supersymmetric extensions of these dualities have also been constructed [17–23].

All these higher-spin/CFT dualities have in common that they relate a weakly coupled higher-spin theory to the weak coupling limit of the corresponding conformal field theory. Furthermore, the conformal field theories of these dualities are typically under relatively good control and the higher-spin theories are commonly considered to be simpler than full-fledged string theory (although this might be debatable). It is therefore reasonable to hope that, besides probing a new regime of AdS/CFT, these dualities may provide us with a promising "laboratory" to study the underlying mechanism of gauge/gravity dualities.

1 See also [15, 16] for the analysis of the asymptotic symmetries of the higher-spin theory which were instrumental for this conjecture.

1.2 OVERVIEW

Broadly speaking, this thesis tries to extract physics out of Vasiliev theory. We study in detail its interactions, spectrum and locality properties. We consider both the three- and four-dimensional case. This is challenging as the theory is formulated in a highly non-standard way.

Free propagation of massless spin-s gauge fields $\phi_{n_1\dots n_s}$ on an AdS background can be described by the *Fronsdal equation*

$$\begin{aligned}\Box\phi_{n_1\dots n_s} - \nabla_{(n_1}\nabla^m\phi_{n_2\dots n_s)m} + \frac{1}{2}\nabla_{(n_1}\nabla_{n_2}\phi_{n_3\dots n_s)m}{}^m \\ - m_s^2\,\phi_{n_1\dots n_s} + 2\Lambda\, g_{(n_1n_2}\,\phi_{n_3\dots n_s)m}{}^m = 0\,.\end{aligned} \tag{1.5}$$

For $s = 2$ we recover the linearized Einstein equations which are obtained by expanding the Einstein equations around an AdS background only keeping linear fluctuations. This leads to an obvious question: can one construct generalizations of the Einstein equations for higher-spin fields? As was mentioned before, this is indeed possible and the corresponding equations are known as *Vasiliev equations*.

Vasiliev theory is in many respects rather exotic. No action, which reduces to the Fronsdal action upon expanding around an (A)dS background, is known[2] and the theory is given in terms of equations of motion. Generically, Vasiliev theory requires the presence of gauge fields with spin $s = 1, 2, 3, \dots, \infty$ and considering any finite subset of them leads to inconsistencies[3]. The theory is based on a generalization of the frame approach to gravity but is crucially formulated using an infinite number of auxiliary fields and coordinates.

Due to its non-standard formulation, a surprising number of basic physical questions about the theory are presently left unanswered. Its field redefinitions are not under control. Naively, Vasiliev theory allows for highly non-local redefinitions which map the theory to a free one. As we will discuss, this makes it difficult to analyze its interactions. For example, so far one cannot unambiguously extract all cubic couplings from Vasiliev equations. Furthermore, three-dimensional Vasiliev theory a priori contains an additional sector of so called twisted fields. It is not known if there exists a consistent truncation of the theory without these additional fields and their presence is surprising as they seem to play no role in the Gaberdiel–Gopakumar duality.

It is these questions that we will study from various viewpoints throughout this thesis. After introducing three-dimensional Vasiliev theory in a (hopefully) pedagogical manner in Chapter 2 and 3, we will turn our attention towards a particularly simple version of this theory. For vanishing matter fields, the three-dimensional theory can be formulated in terms of a Chern–Simons action and in this special

2 There exist however actions whose Euler–Lagrange equations are the Vasiliev equations. See [24] for a recent review and references therein.

3 To be more precise, there exists a truncation of Vasiliev theory to even spins only. But also this truncation obviously contains an infinite amount of higher-spin fields.

case it is possible to truncate the infinite tower of higher-spin fields to a finite subset. In Chapter 4, we will study a formulation of this theory in terms of metric-like variables as this may allow for a more geometrical understanding of the higher-spin theory.

In Chapter 5, we will then study the three-dimensional theory for non-vanishing matter fields. In this case the only known description of the theory is given in terms of Vasiliev equations. We will analyze them to second order in perturbations around an AdS_3 background. We will show that there exists a field frame in which all twisted fields can consistently be set to zero (up to this order in perturbation theory). This provides non-trivial evidence for a truncation of Vasiliev theory without these additional fields. We will then study the couplings of the matter and (higher-spin) gauge fields. From general arguments, the structure of these interactions is known up to field redefinitions and only involves a finite number of derivatives. We want to fix their coefficients by analyzing Vasiliev theory. However, the interactions which we obtain involve an infinite number of derivatives and are therefore potentially non-local. Since the field redefinitions of Vasiliev theory are not under control, this confronts us with a serious challenge.

At this stage, one may wonder if the fact that we can remove any interaction by a field redefinition is a pathology of the three-dimensional theory. After all the (higher-spin) gauge fields do not propagate in three dimensions and so one has to be careful with what is meant by an interacting theory as the gauge degrees of freedom only reside on the boundary. As we will however establish in Chapter 6, an analysis of four-dimensional Vasiliev theory leads to analogous challenges.

In Chapter 7, we will therefore study the allowed class of field redefinitions in more detail. We will make a concrete proposal for a set of permissible field redefinitions and discuss non-trivial evidence for this conjecture. This proposal will confront us with the puzzling observation that the interactions of Vasiliev theory contain divergent couplings.

* * *

Chapter 2 and 3 and the first half of Chapter 6 review the basics of three and four-dimensional Vasiliev theory. Special emphasis is put on presenting this material in an accessible way.

Chapter 4 closely follows the publication [25] with my supervisor Stefan Fredenhagen. The discussion is slightly extended by a section which makes contact with the three-dimensional Vasiliev theory of the previous chapter. A few of the more technical discussions are omitted.

Chapter 5 is based on the publication [26] together with Gustavo Lucena Gómez, Evgeny Skvortsov and Massimo Taronna. Two important results contained in this paper are not discussed. Namely, we omit the presentation of a complete cubic action for three-dimensional Vasiliev theory obtained by symmetry arguments. Secondly, we will not discuss a cohomological analysis of the interaction terms of Vasiliev theory.

These results are not of crucial importance for the main narrative of this thesis and omitting them allows for a streamlined account of our analysis.

The second half of Chapter 6 is based on work [27] with Nicolas Boulanger, Evgeny Skvortsov and Massimo Taronna. Our discussion roughly follows the presentation therein. We will not discuss the deformation of second order Weyl tensors.

The results of Chapter 7 were also contained in [27] but we outline a different derivation of them. This method is complementary to the one used in [27] and therefore provides a cross-check of our results.[4]

The appendices summarize more technical aspects of higher-spin theories, such as the oscillator representation of the higher-spin algebra, manifest local Lorentz symmetry and an overview of the σ_--cohomology. This material is well established but I tried to present it in, at least to my taste, more pedagogical manner.

1.3 SOME ADVICE ON READING THIS THESIS

The author certainly hopes that some readers will find it worthwhile to read this thesis from cover to cover but there are also alternative approaches depending on the material the reader wants to study: for those interested in learning three-dimensional Vasiliev theory, we recommend reading Chapter 2 and Chapter 3. The reader can then skip to Section 6.1-6.8 of Chapter 6 for an introduction to four-dimensional Vasiliev theory. The discussion of the divergent couplings in Chapter 7 is mostly self-contained and, taking the point of view that Vasiliev theory provides a "black box" to extract the second-order equations of motion, can hopefully be read without any detailed understanding of Vasiliev theory.

4 I want to thank Evgeny Skvortsov who was of critical importance in developing this method.

Part I

FREE EQUATIONS IN ARBITRARY DIMENSION

2 FREE HIGHER-SPIN THEORY

In this section, free equations of motion for higher-spin theory will be discussed both in terms of metric-like and frame-like variables. The latter will be of central importance for Vasiliev theory which we will introduce in Chapter 3.

2.1 FREE METRIC-LIKE THEORY

We will first focus on the case of a flat Minkowski spacetime and then generalize the discussion to AdS backgrounds. Our presentation will closely follow [28].

2.1.1 *Minkowski Background*

The free equations of motion of a massless spin-1 gauge field in d-dimensional Minkowski spacetime are given by

$$\Box A_n - \partial_n \partial^m A_m = 0\,. \tag{2.1}$$

This equation is invariant under the spin-1 gauge transformation

$$\delta A_n = \partial_n \xi(x)\,, \tag{2.2}$$

where we denote spacetime indices by $m, n, \ldots$ and spacetime coordinates by x^m.

Similarly, the following free equations of motion for the spin-2 field h_{nn} are obtained by linearizing Einstein's equations around a Minkowski background

$$R_{nn} := \Box h_{nn} - \partial_n \partial^m h_{mn} + \partial_n \partial_n h^m{}_m = 0\,, \tag{2.3}$$

which are invariant under spin-2 gauge transformations

$$\delta h_{nn} = \partial_n \xi_n(x)\,. \tag{2.4}$$

Here we use the following notation: indices on the same level and denoted by the same letter are understood to be symmetrized by adding all necessary permutations without any additional factors, e. g. the term $\partial_n \xi_n$ denotes $\partial_{n_1} \xi_{n_2} + \partial_{n_2} \xi_{n_1}$. In the following we will further ease notation by denoting a fully symmetric rank-s tensor by $T_{n(s)}$, e. g. $h_{n(2)}$ is the same as h_{nn}. While this notation might seem contrived for the case of spin 2, it will turn out to be very efficient for the case of generic spin.

The spin-1 and spin-2 equations of motion can be generalized for a spin-s field by the *Fronsdal equation* [29]

$$F_{n(s)} := \Box\phi_{n(s)} - \partial_n\partial^m\phi_{mn(s-1)} + \partial_n\partial_n\phi_{n(s-2)m}{}^m = 0\,. \qquad (2.5)$$

This equation is invariant under the following spin-s gauge transformation

$$\delta\phi_{n(s)} = \partial_n\xi_{n(s-1)}(x)\,, \qquad \xi_{n(s-2)m}{}^m = 0\,. \qquad (2.6)$$

The trace constraint on the gauge parameter is crucial for gauge invariance of the *Fronsdal tensor* $F_{n(s)}$ as it transforms by

$$\delta F_{n(s)} \sim \partial_n\partial_n\partial_n\xi_{n(s-3)m}{}^m\,. \qquad (2.7)$$

Note that the trace constraint has no effect for $s = 1, 2$.

The free dynamics of a spin-s gauge field can be described by the following *Fronsdal action*

$$S = \frac{1}{2}\int \mathrm{d}^d x\, \phi_{n(s)}\mathcal{F}^{n(s)}\,, \qquad (2.8)$$

where we defined

$$\mathcal{F}_{n(s)} := F_{n(s)} - \frac{1}{2}\eta_{nn}F_{n(s-2)m}{}^m\,. \qquad (2.9)$$

This tensor is the spin-s generalization of the linearized Einstein tensor. By partial integration, it can be easily seen that gauge invariance of the Fronsdal action

$$\delta S = -s\int \mathrm{d}^d x\, \xi_{n(s-1)}\partial_m\mathcal{F}^{n(s-1)m} \overset{!}{=} 0\,, \qquad (2.10)$$

requires the following Bianchi identity to hold

$$\partial^m\mathcal{F}_{mn(s-1)} \sim \partial_n\partial_n\partial_n\phi_{n(s-4)mp}{}^{mp} \overset{!}{=} 0\,, \qquad (2.11)$$

which imposes a double-tracelessness constraint on the Fronsdal field, i.e. $\phi^{mn}{}_{mn\dots} = 0$.

The equations of motion are easily derived from the Fronsdal action by using the fact that the action is symmetric in its spin-s fields and read

$$\mathcal{F}_{n(s)} = 0\,. \qquad (2.12)$$

However, double-tracelessness of the Fronsdal field implies $F^{mn}{}_{mn\dots} = 0$ as can be seen from the explicit form of the Fronsdal tensor as given in (2.5). This observation implies that[1]

$$\mathcal{F}_{n(s-1)m}{}^m \sim F_{n(s-1)m}{}^m\,. \qquad (2.13)$$

1 Here we assume that $d \geq 3$ as the proportionality factor vanishes for $d = 2$ and $s = 2$.

Therefore, the equation of motion (2.12) implies that $F^n{}_{n\dots} = 0$ and thus by the explicit form (2.9) of $\mathcal{F}_{n(s)}$ it follows that

$$\mathcal{F}_{n(s)} = 0 \qquad \Longleftrightarrow \qquad F_{n(s)} = 0\,. \tag{2.14}$$

The equation of motion derived from the Fronsdal action (2.12) is therefore completely equivalent to the Fronsdal equation (2.5).

It can be checked that solutions to the Fronsdal equation carry the correct number of propagating degrees of freedom of a massless spin-s field [30], i.e. the degrees of freedom of a completely symmetric and traceless rank-s tensor in $d-2$ dimensions. To show this it is essential that the Fronsdal field is double-traceless. In particular in four dimensions, there are two propagating degrees of freedom corresponding to the two helicities of a massless spin-s field. In three dimensions, the spin-s field does not carry any propagating degrees of freedom.[2] It is however important to stress that this counting only includes local and not global degrees of freedom.

2.1.2 *AdS Background*

One can also formulate a free theory for spin-s gauge fields on AdS$_d$ backgrounds with metric g_{nm}. This will be of crucial importance in the following as we will be interested in solutions of Vasiliev theory with negative cosmological constant $\Lambda < 0$. At the free level, spin-s gauge fields on AdS$_d$ obey the following gauge transformations

$$\delta\phi_{n(s)} = \nabla_n \xi_{n(s-1)}\,, \qquad g^{op}\xi_{n(s-3)op} = 0\,, \tag{2.15}$$

where ∇_n denotes the covariant derivative with respect to the AdS$_d$ background.

The main complication in formulating such a theory as compared to Minkowski space is that the covariant derivatives do not commute

$$[\nabla_n, \nabla_m]V_p = \Lambda\left(g_{np}V_m - g_{mp}V_n\right)\,. \tag{2.16}$$

The gauge invariant generalization of the Fronsdal tensor of Minkowski space is then given by

$$\begin{aligned}\hat{F}_{n(s)} := \Box\phi_{n(s)} - \nabla_n\nabla^m\phi_{mn(s-1)} + \frac{1}{2}\nabla_n\nabla_n\phi_{n(s-2)m}{}^m \\ - m_s^2\,\phi_{n(s)} + 2\Lambda\, g_{nn}\,\phi_{n(s-2)m}{}^m = 0\,.\end{aligned} \tag{2.17}$$

The first line is the straightforward generalization of (2.5) for AdS space.[3] The second line subtracts all terms produced by the commuta-

2 This is only true for $s > 1$. Maxwell theory in three dimensions describes a propagating scalar degree of freedom.

3 The relative factor of $\frac{1}{2}$ in the third term as compared to (2.5) is due the chosen symmetrization conventions. In flat space, one needs $\binom{s}{2} = \frac{s(s-1)}{2}$ terms for symmetrization while on AdS $s(s-1)$ terms are required.

tor (2.16) and thereby ensures gauge invariance for a particular value of the mass term

$$m_s^2 = -2\Lambda(s-1)(d+s-3)\,. \tag{2.18}$$

Note that gauge invariance on AdS requires the presence of a mass term due to the non-commutativity of the covariant derivatives while on a Minkowski background such a term would break gauge invariance.

One can then construct an action in complete analogy to the flat case which reads

$$S = \frac{1}{2}\int \mathrm{d}^d x\sqrt{|g|}\,\phi_{n(s)}\hat{\mathcal{F}}^{n(s)}\,, \tag{2.19}$$

where we defined

$$\hat{\mathcal{F}}_{n(s)} := \hat{F}_{n(s)} - \frac{1}{2}g_{nn}\hat{F}_{n(s-2)m}{}^{m}\,. \tag{2.20}$$

Gauge invariance is again enforced by requiring double-tracelessness for the Fronsdal field, i.e. $g^{nm}g^{op}\phi_{nmop\ldots} = 0$. Let us mention that this discussion also holds for the case of dS backgrounds with $\Lambda > 0$ but we will not consider such backgrounds in the following. Thus all maximally symmetric spacetimes allow for a consistent formulation of Fronsdal equations. While maximal symmetry of spacetime backgrounds is therefore a sufficient condition for the consistency of Fronsdal equations, a necessary and sufficient condition is not known and is certainly not given by maximal symmetry. For example, it has been recently shown that product manifolds of the form $\mathrm{AdS}_p \times \mathrm{S}^q$ for arbitrary $p, q \in \mathbb{N}$ also admit consistent propagation of massless higher-spin gauge fields provided that the radii of the two product manifolds are the same [31].

2.2 FREE FRAME-LIKE THEORY

In this section, the frame-like formalism for free higher-spin gauge theories will be developed. This formalism will be an important ingredient of the non-linear Vasiliev theory. We will mainly follow the presentation in [28] and [32]. The seminal papers on this subject are [33–35].

2.2.1 *Minkowski Background*

In the frame-like formulation of gravity, a Minkowski background can be described in terms of a background vielbein, $\bar{e}^a = \bar{e}^a_m dx^m$, and spin-connection, $\bar{\omega}^{a,b} = \bar{\omega}^{a,b}_m dx^m$, where $a, b, c, \ldots$ denote local Lorentz indices. Let us recall for the spin-2 case how the free theory can be described in terms of the frame-like variables. The vielbein and spin-connection transform by

$$\delta e^a = \nabla\epsilon^a + \bar{e}_b\,\xi^{a,b}\,, \tag{2.21}$$

$$\delta\omega^{a,b} = \nabla\xi^{a,b}\,, \tag{2.22}$$

where $\nabla t^a = \mathrm{d}t^a - \bar{\omega}^a{}_b t^b$ is the Lorentz-covariant derivative and the gauge parameter of the local Lorentz rotations is antisymmetric, i.e. $\xi^{a,b} = -\xi^{b,a}$. The gauge parameter ϵ^a parametrizes local translations and is closely related to diffeomorphisms as we will see momentarily. Choosing coordinates such that $\bar{e}^a_m = \delta^a_m$ and $\bar{\omega}^{a,b}_m = 0$ allows one to straightforwardly convert spacetime indices to local Lorentz indices, e.g.

$$\delta e_{nb}\, \bar{e}^n_a = \delta e_{nb}\, \delta^n_a =: \delta e_{ab}\,, \tag{2.23}$$

where the inverse background vielbein $\bar{e}^n_a$ is defined by $\bar{e}^n_a \bar{e}^a_m = \delta^n_m$. The symmetric part of the vielbein transforms by (2.21) as

$$\delta e_{(ab)} = \nabla_{(a}\epsilon_{b)} \tag{2.24}$$

and can therefore be identified with h_{ab} in (2.3) while ϵ_a parameterizes diffeomorphisms. On the other hand the antisymmetric part obeys

$$\delta e_{[ab]} = \nabla_{[a}\epsilon_{b]} - \xi_{a,b}\,. \tag{2.25}$$

Thus, by an appropriate choice for $\xi_{a,b}$, one can always gauge away the antisymmetric component of the vielbein $e_{[ab]}$ as $\xi_{a,b}$ enters the transformation rule algebraically. One can then construct gauge invariant curvatures

$$R^a := \nabla e^a + \bar{e}_b \wedge \omega^{a,b}\,, \tag{2.26}$$

$$R^{a,b} := \nabla \omega^{a,b}\,. \tag{2.27}$$

The torsion constraint

$$R^a = 0 \tag{2.28}$$

can be used to express the spin-connection $\omega^{a,b}$ in terms of the vielbein e^a. Upon plugging this solution into (2.27), one can rederive the linearized Riemann tensor by identifying the symmetric part of the vielbein with h_{mn}. The linearized Einstein equations are then equivalent to

$$\bar{e}^n_a\, R^{a,b}_{nm} = 0\,, \tag{2.29}$$

where we have used $R^{a,b} = R^{a,b}_{nm}\, \mathrm{d}x^n \wedge \mathrm{d}x^m$.

This analysis can be generalized to higher-spins. For this purpose, a generalized vielbein is introduced

$$e^{a(s-1)} = e^{a(s-1)}_m dx^m\,, \qquad\qquad e^{a(s-3)b}{}_b = 0\,, \tag{2.30}$$

which transforms as follows

$$\delta e_{a(s-1)} = \nabla \epsilon_{a(s-1)} + \bar{e}^b\, \xi_{a(s-1),b}\,. \tag{2.31}$$

The gauge parameter ξ has vanishing projection on its fully symmetric part

$$\xi_{a(s-1),a} = 0\,, \tag{2.32}$$

and all gauge parameters are completely traceless[4]

$$\epsilon^{b}{}_{ba(s-3)} = 0\,, \qquad\qquad \xi^{b}{}_{ba(s-3),c} = 0\,. \tag{2.33}$$

Using again coordinates for which $\bar{e}^{a}_{m} = \delta^{a}_{m}$ one observes that the fully symmetric part of the generalized vielbein, $\phi_{a(s)} := e_{n\,a(s-1)}\bar{e}^{n}_{a}$, transforms by (2.32) as

$$\delta\phi_{a(s)} = \nabla_a \epsilon_{a(s-1)}\,, \qquad\qquad \text{with } \epsilon^{b}{}_{ba(s-3)} = 0\,, \tag{2.34}$$

and therefore is to be identified with the Fronsdal field of (2.5). This statement obviously generalizes for other choices of coordinates and one obtains

$$\phi_{n(s)} = e^{a(s-1)}_{n}\bar{e}_{n\,a}\ldots\bar{e}_{n\,a}\,. \tag{2.35}$$

From this equation, it can be seen that tracelessness of the generalized vielbein imposes double-tracelessness for the Fronsdal field. All other components of the generalized vielbein can again be gauged away by appropriate choice of $\xi^{a(s-1),b}$ which enters the transformation rule for the remaining components algebraically. One then introduces the analog of the spin-connection for gauge transformations parameterized by the $\xi^{a(s-1),b}$ parameter

$$\omega^{a(s-1),b} = \omega_{n}^{a(s-1),b}dx^{n}\,, \qquad \omega^{a(s-1),a} = 0\,, \qquad \omega_{b}{}^{ba(s-3),c} = 0\,.$$

The straightforward higher-spin generalization of curvature arising in the spin-2 torsion constraint is

$$R^{a(s-1)} := \nabla e^{a(s-1)} - \bar{e}_{b} \wedge \omega^{a(s-1),b}\,. \tag{2.36}$$

However, this curvature is invariant under

$$\delta\omega^{a(s-1),b} = \nabla\xi^{a(s-1),b} - \bar{e}_{c}\,\xi^{a(s-1),bc}\,, \tag{2.37}$$

which contains an additional gauge parameter $\xi^{a(s-1),bc}$ which obeys

$$\xi^{a(s-1),bc} = \xi^{a(s-1),cb}\,, \qquad \xi^{a(s-1),ab} = 0\,, \qquad \xi_{b}{}^{ba(s-3),c(2)} = 0\,.$$

It is natural to choose to add a corresponding gauge field $\omega^{a(s-1),b(2)}$ for this additional algebraic gauge symmetry. We will discuss the reason for this choice in detail in Section 2.2.3. This extra field $\omega^{a(s-1),b(2)}$ is then also part of the curvature

$$R^{\alpha(s-1),b} := \nabla\omega^{a(s-1),b} - \bar{e}_{c} \wedge \omega^{a(s-1),bc}\,. \tag{2.38}$$

4 We note that (2.32) and (2.33) also imply $\xi^{b}{}_{a(s-2),b} = 0$.

By analogous arguments as above, we again see that this has an additional gauge symmetry parameterized by $\xi^{a(s-1),b(3)}$.

We can then iterate this procedure to obtain

$$R^{a(s-1),b(t)} := \nabla\omega^{a(s-1),b(t)} - \bar{e}_c \wedge \omega^{a(s-1),b(t)c}\,, \tag{2.39}$$

which is invariant under

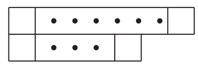
$$\delta\omega^{a(s-1),b(t)} = \nabla\xi^{a(s-1),b(t)} - \bar{e}_c\,\xi^{a(s-1),b(t)c}\,, \tag{2.40}$$

where all tensors of the form $T^{a(s-1),b(t)}$ obey[5]

$$T^{a(s-1),ab(t-1)} = 0\,, \qquad\qquad T_b{}^{ba(s-3),c(t)} = 0\,. \tag{2.41}$$

Tensors of this type transform in a representation of the Lorentz algebra corresponding to the Young diagram

s-1



t

For a hands-on introduction to Young diagrams and their application in higher-spin theory see Appendix E of [28] and Section 4 of [36].

This process terminates at $t = s$ as the corresponding extra field vanishes by symmetry, i.e. $\omega^{a(s-1),b(s)} \equiv 0$, and one obtains

$$R^{a(s-1),b(s-1)} := \nabla\omega^{a(s-1),b(s-1)}\,, \tag{2.42}$$

which is invariant under

$$\delta\omega^{a(s-1),b(s-1)} = \nabla\xi^{a(s-1),b(s-1)}\,. \tag{2.43}$$

Notice that for $s = 2$ the expression (2.42) is the linearized Riemann curvature (2.27). These curvatures can be used to construct free frame-like equations of motion as we will discuss in Section 2.2.3.

Let us summarize the results discussed in this section using notation that will be useful to generalize our analysis to AdS backgrounds. We define

$$\omega^{a(s-1)} := e^{a(s-1)} \qquad \text{and} \qquad \xi^{a(s-1)} := \epsilon^{a(s-1)}\,. \tag{2.44}$$

The curvatures derived above then read

$$\boxed{\begin{aligned} R^{a(s-1),b(t)} &:= \nabla\omega^{a(s-1),b(t)} + \sigma_-(\omega)^{a(s-1),b(t)}\,, && (2.45\text{a})\\ \delta\omega^{a(s-1),b(t)} &= \nabla\xi^{a(s-1),b(t)} + \sigma_-(\xi)^{a(s-1),b(t)}\,, && (2.45\text{b}) \end{aligned}}$$

where $0 \leq t \leq s-1$ and we defined

$$\sigma_-(T)^{a(s-1),b(t)} = \begin{cases} \bar{e}_c \wedge T^{a(s-1),b(t)c} & 0 \leq t < s-1 \\ 0 & t = s-1 \end{cases}\,, \tag{2.46}$$

with $T^{a(s-1),b(t+1)}$ denoting either the gauge parameters $\xi^{a(s-1),b(t+1)}$ or fields $\omega^{a(s-1),b(t+1)}$. The operator σ_- is obviously nilpotent $\sigma_-^2 = 0$. Using this property, it is straightforward to show that (2.45a) is invariant under (2.45b) by also taking into account that

$$\nabla^2 = 0 \tag{2.47}$$

on a Minkowski background and

$$\{\nabla, \sigma_-\} = 0\,, \tag{2.48}$$

which follows from the vielbein postulate $\nabla\bar{e}^a = 0$.

---

5 From these properties it can also be shown that all other traces vanish, i.e. $T_b{}^{a(s-2),bc(t-1)} = 0$ and $T^{a(s-1),bc(t-1)}{}_b = 0$. See Appendix E of [28] for a detailed derivation.

2.2.2 *AdS Background*

The results obtained in the last section can be generalized to AdS backgrounds. The main complication in this arises from the fact that $\nabla^2 \sim \Lambda\, \bar{e} \wedge \bar{e} \neq 0$ which implies that the naive generalization of the curvatures (2.45a) is no longer invariant under the gauge transformations (2.45b) for an AdS background. In order to construct gauge invariant linearized curvatures, one writes down the most general structure which is compatible with local Lorentz covariance and has the correct flat limit

$$\boxed{\begin{aligned} R^{a(s-1),b(t)} &:= \nabla \omega^{a(s-1),b(t)} + \sigma(\omega)^{a(s-1),b(t)} \,, \qquad (2.49a)\\ \delta\omega^{a(s-1),b(t)} &= \nabla \xi^{a(s-1),b(t)} + \sigma(\xi)^{a(s-1),b(t)} \,, \qquad (2.49b) \end{aligned}}$$

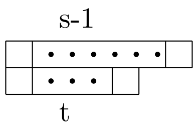
where we defined

$$\sigma(T)^{a(s-1),b(t)} = \sigma_-(T)^{a(s-1),b(t)} + \beta\, \Pi \left\{ \bar{e}^b \wedge T^{a(s-1),b(t-1)} \right\} \,, \tag{2.50}$$

where $0 \leq t \leq s-1$ and σ_- is given as before by (2.46). The parameter β obeys

$$\beta \to 0 \;\; \text{for } \Lambda \to 0 \,, \tag{2.51}$$

as to obtain the correct flat limit. The projector Π projects on a tensor with symmetry corresponding to the following Young diagram



Invariance of (2.49a) under (2.49b) requires

$$\nabla^2 + \sigma^2 = 0 \,, \tag{2.52}$$

as can be easily derived by using $\{\nabla, \sigma\} = 0$, which again follows from the vielbein postulate $\nabla \bar{e}^a = 0$. The constant β can therefore be fixed by (2.52). Explicit expressions for Π and β are rather lengthy and can be found in [35]. This shows that requiring both gauge invariance for non-vanishing cosmological constant and the correct flat limit fixes the linearized curvatures completely.

As we will discuss, Vasiliev theory is formulated by mapping local Lorentz to spinorial indices. One advantage of doing so lies in the fact that the projector Π becomes particularly simple.

Let us close this section by mentioning that the discussion above also applies for dS backgrounds upon flipping the sign of the cosmological constant Λ.

2.2.3 *Recovering Fronsdal*

In this section, we will show that the Fronsdal equation (2.5) can be obtained from the frame-like theory following Section 5.2 of [28]. The equations of motion for a Minkowski background are given by

$$R^{a(s-1)} = 0\,, \tag{2.53a}$$
$$\bar{e}^{am}\bar{e}^{n}_{b}\, R_{mn}{}^{a(s-1),b} = 0\,, \tag{2.53b}$$

where we have used $R^{a(s-1),b} = R_{mn}{}^{a(s-1),b}\, \mathrm{d}x^m \wedge \mathrm{d}x^n$. Let us recall the relevant curvatures of (2.45a) for convenience

$$R^{a(s-1)} = \nabla e^{a(s-1)} + \bar{e}_c \wedge \omega^{a(s-1),c}\,, \tag{2.54}$$
$$R^{a(s-1),b} = \nabla \omega^{a(s-1),b} + \bar{e}_c \wedge \omega^{a(s-1),bc}\,. \tag{2.55}$$

As we stressed before, it was only a choice to introduce the extra fields $\omega^{a(s-1),b(t)}$ with $t > 1$. As we will show in the following, they are not required to reproduce the Fronsdal equation (2.5).

For flat coordinates with $\nabla = \mathrm{d}$ and $\bar{e}^n_a = \delta^n_a$, these curvatures can be rewritten as

$$R^{a(s-1)}_{nm}\bar{e}^{n\,c}\bar{e}^{m\,d} = \partial^c e^{a(s-1)|d} + \omega^{a(s-1),c|d} - c \leftrightarrow d\,, \tag{2.56}$$
$$R^{a(s-1),b}_{nm}\bar{e}^{n\,c}\bar{e}^{m\,d} = \partial^c \omega^{a(s-1),b|d} + \omega^{a(s-1),bc|d} - c \leftrightarrow d\,, \tag{2.57}$$

where we used the notation $e^{a(s-1)|b} := e^{a(s-1)}_n \bar{e}^{nb}$ and similarly for all other fields. We now project on the second frame-like equation of motion (2.53b) by contracting the corresponding curvature (2.57) with η_{bd} and symmetrizing with respect to $c \leftrightarrow a$. As we will show, the resulting expression indeed corresponds to the Fronsdal equation

$$F^{a(s)} \stackrel{!}{=} \partial_b \omega^{a(s-1),b|a} - \partial^a \omega^{a(s-1),b|}{}_b = 0\,. \tag{2.58}$$

It is important to note that the extra field $\omega^{a(s-1),bc|d}$ dropped out of this equation and therefore could have been omitted in the first place.

To show that (2.58) indeed coincides with the Fronsdal equation (2.5), we symmetrize (2.56) with respect to $c \leftrightarrow a$. Since the resulting expression has to vanish by the torsion constraint, we obtain

$$\omega^{a(s-1),b|a} = \partial^a e^{a(s-1)|b} - \partial^b e^{a(s-1)|a}\,. \tag{2.59}$$

Inserting this result in (2.58) and identifying $\phi^{a(s)} = e^{a(s)|a}$, one indeed recovers the Fronsdal equation

$$\Box\phi^{a(s)} - \partial^a\partial_b\phi^{ba(s-1)} + \partial^a\partial^a\phi^{a(s-2)b}{}_b = 0\,. \tag{2.60}$$

This discussion might leave the reader wondering why one chooses to add the extra fields since the correct metric-like free equations can be obtained without introducing them. As we will discuss in detail, Vasiliev equations are based on a different formulation of the equations of motion than the one presented above. This *unfolded formulation* of the free equations is given in terms of all curvatures $R^{a(s),b(t)}$ and therefore by extension requires the extra fields. This indicates that they are required for consistency at the non-linear order.

Since Vasiliev theory is formulated in terms of the unfolded equations of motion, we will only briefly comment on the following aspects of the free equations given above:

- From our discussion, it is not entirely obvious that the system of equations (2.53) is equivalent to Fronsdal equation since we have considered a particular symmetrization of the generalized torsion constraint. This can however be shown [33].

- Analogous equations of motions can be formulated for an AdS background [34, 35] but we will not discuss them.

- A free action for these frame-like equations of motion for both Minkowski and (A)dS-backgrounds is known [33–35]. For non-vanishing cosmological constant, this action has remarkable similarities with the MacDowell-Mansouri-Stelle-West gravity action (see Chapter 11 of [28] for a pedagogical introduction and references therein).

- In three dimensions, the frame-like Fronsdal equation (2.53b) is equivalent to $R^{a(s-1),b} = 0$. This statement follows by similar reasons as in the case of three-dimensional gravity. We will discuss the three-dimensional case in detail in the next chapter.

We will however discuss the equivalence of the unfolded formulation and Fronsdal equations in detail in Chapter 6.

Part II

THREE DIMENSIONS

3 THREE-DIMENSIONAL VASILIEV THEORY

In this chapter, we will give an introduction to three-dimensional Vasiliev theory. *This theory is often also referred to as Prokushkin–Vasiliev theory but we will not use this term in the following.* The reasons for this are two-fold: for technical simplicity, we will mostly consider the undeformed theory which is based on the gauge algebra $\mathfrak{hs}(\lambda)$ with $\lambda = \frac{1}{2}$. This theory historically preceded the construction for generic values of λ and was first given by Vasiliev in [37]. Furthermore, three- and four-dimensional higher-spin theories will be discussed in this thesis and it is therefore useful to refer to both of them as Vasiliev theories.

This chapter consists of two parts:

- Section 3.1 gives an introduction to the free equations of motion in unfolded form. This form of the equations is important as it forms the starting point for Vasiliev's construction of non-linear equations.
- Section 3.2 will then discuss Vasiliev equations. The free unfolded equations of motions will be derived by linearizing Vasiliev theory.

Seminal publications on this subject are [37] and [2]. The former provides a comparatively accessible and succinct introduction to the subject. In learning Vasiliev theory, the author also found the Appendix A of [38] useful.

3.1 FREE THEORY

In the following, we will first review some aspects of three-dimensional gravity and then construct a particularly useful realization of the AdS_3-isometry algebra which will allow us to introduce the unfolded formulation of free equations for both higher-spin and matter fields on an AdS_3-background. We will then discuss the higher-spin algebra and close by introducing twisted fields.

3.1.1 *A Lightning Review of Three-Dimensional Gravity*

Einstein field equations are given by

$$R_{mn} - \frac{1}{2} R\, g_{mn} + \Lambda\, g_{mn} = 0\,. \tag{3.1}$$

The Riemann curvature tensor can be decomposed as follows

$$R_{mnrp} = W_{mnrp} - g_{m[r} R_{p]n} + \frac{1}{2} R\, g_{m[r} g_{p]n}\,. \tag{3.2}$$

The Weyl tensor is identically zero due to the fact that tensors associated with Young diagrams with more than three boxes in the first two columns vanish in three dimensions. See [28] for more details.

In three dimensions, the completely traceless Weyl tensor W_{mnrp} is identically zero. This implies that the theory does not contain propagating degrees of freedom. By contracting Einstein equations with g^{mn}, one obtains

$$R = \frac{2d}{d-2}\Lambda = 6\,\Lambda\,. \tag{3.3}$$

and inserting this result in the Einstein equations leads to

$$R_{nm} = \frac{2}{d-2}\Lambda\, g_{nm} = 2\,\Lambda\, g_{nm}\,. \tag{3.4}$$

Together with (3.2) for vanishing Weyl tensor, this implies

$$R_{mnrp} = \Lambda\left(g_{mr}g_{pn} - g_{mp}g_{rn}\right)\,. \tag{3.5}$$

Therefore, all three-dimensional spacetime solutions have constant curvature and are thus locally diffeomorphic to the maximally symmetric solution. This statement does not, however, hold globally. For example, there exist BTZ black hole solutions for the case of negative cosmological constant [39]. In the frame-like language, the constant curvature condition (3.5) is equivalent to

$$F^{ab} := R^{ab} - \Lambda\, e^a \wedge e^b = 0\,, \tag{3.6a}$$

$$F^{a} := \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0\,, \tag{3.6b}$$

where we defined $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$. In order to show that this set of equations is equivalent to (3.5), one has to use the torsion constraint (3.6b) to express the spin-connection in terms of the vielbein, i.e. $\omega(e)$, and plug the result into (3.6a). The equivalence to (3.5) then follows from

$$R(\omega(e))^{ab}{}_{nm} = R_{mnrp}\, e^{r\,a} e^{p\,b} \qquad \text{and} \qquad g_{mn} = \eta_{ab}\, e^a_n e^b_m\,, \tag{3.7}$$

where the inverse vielbein e^n_a is defined by $e^n_a e^b_n = \delta^a_b$. One can rewrite (3.6) in a more compact form

$$\mathrm{d}\Omega + \Omega \wedge \Omega = 0 \qquad \Leftrightarrow \qquad \frac{1}{2}F^{ab}L_{ab} + F^a P_a = 0\,, \tag{3.8}$$

where the connection Ω is defined by

$$\Omega = e^a P_a + \frac{1}{2}\omega^{ab}L_{ab}\,, \tag{3.9}$$

and P_a, L_{ab} are generators of the isometry algebra of the maximally symmetric solution obeying the following commutation relations

$$[P_a, P_b] = -\Lambda\, L_{ab}\,, \tag{3.10a}$$

$$[P_a, L_{bc}] = \eta_{ab}\, P_c - \eta_{ac}\, P_b\,, \tag{3.10b}$$

$$[L_{ab}, L_{cd}] = L_{ad}\,\eta_{bc} - L_{bd}\,\eta_{ac} - L_{ac}\,\eta_{bd} + L_{bc}\,\eta_{ad}\,. \tag{3.10c}$$

Up to this point, our discussion will also go through for arbitrary dimensions if one restricts to solutions for which the constant curvature condition (3.5) holds (which is always the case for $d = 3$). However, in three dimensions, the equations of motion (3.8) can be derived from a Chern–Simons action

$$S = \frac{1}{16\pi} \int \mathrm{tr} \left(\Omega \wedge \mathrm{d}\Omega + \frac{2}{3} \Omega \wedge \Omega \wedge \Omega \right) . \tag{3.11}$$

Up to boundary terms, this action is the frame-like Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int \epsilon_{abc} \left(e^a \wedge R^{bc} - \frac{\Lambda}{3}\, e^a \wedge e^b \wedge e^c \right) , \tag{3.12}$$

if one chooses the following invariant bilinear form[1]

$$\mathrm{tr}\,(P_a L_{bc}) = \frac{1}{G}\, \epsilon_{abc} , \quad \mathrm{tr}\,(P_a P_b) = 0 , \quad \mathrm{tr}\,(L_{ab} L_{cd}) = 0 . \tag{3.13}$$

Let us stress that this bilinear form only exists in three dimensions. There exists also another bilinear form, which is well-defined for arbitrary dimensions, but would not lead to the Einstein-Hilbert action [40].

This construction works for arbitrary cosmological constant. Specializing further on the case of negative cosmological constant, we define the AdS radius l as $\Lambda = -\frac{1}{l^2}$. By dualizing $L_a = -\frac{1}{2}\epsilon_{abc} L^{bc}$ we can rewrite the AdS_3 isometry algebra using the following generators

$$K_a = \frac{1}{2}(L_a + l P_a) , \qquad \tilde{K}_a = \frac{1}{2}(L_a - l P_a) . \tag{3.14}$$

They obey the commutation relations

$$[K_a, K_b] = \epsilon_{abc} K^c , \quad [\tilde{K}_a, \tilde{K}_b] = \epsilon_{abc} \tilde{K}^c , \quad [K_a, \tilde{K}_b] = 0 , \tag{3.15}$$

showing that the AdS_3-isometry algebra splits as $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

One can now decompose the connection Ω in terms of these generators

$$\Omega = A^a K_a + \tilde{A}^a \tilde{K}_a , \tag{3.16}$$

with

$$A^a = \omega^a + \frac{1}{l} e^a \qquad \text{and} \qquad \tilde{A}^a = \omega^a - \frac{1}{l} e^a , \tag{3.17}$$

where $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$.

The action (3.11) with bilinear form (3.13) can be rewritten as the difference of two Chern-Simons theories

$$S = S_{CS}\,[A] - S_{CS}\,[\tilde{A}] \ , \tag{3.18}$$

1 The factor of $\frac{1}{G}$ ensures that the dimensions of both side of the equation balance ($[P_a] = \frac{1}{\mathrm{length}}$, $[L_a] = 0$ and $[G] = \frac{1}{\mathrm{length}}$).

with the Chern-Simons action given by

$$S_{CS}\left[A\right] = \frac{\hat{k}}{4\pi}\int \mathrm{tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) . \tag{3.19}$$

A similar construction holds for dS_3. However, one then has $A^a = \omega^a + \frac{i}{|l|}e^a$ and $\tilde{A}^a = (A^a)^*$. Hence, the gauge group for positive cosmological constant is $\mathfrak{sl}(2,\mathbb{C})$ as opposed to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ for negative cosmological constant. For more details see [40, 41].

The bilinear form (3.13) implies

$$\mathrm{tr}\left(K_a K_b\right) = \tfrac{1}{2}\eta_{ab}\,, \tag{3.20a}$$
$$\mathrm{tr}\left(\tilde{K}_a \tilde{K}_b\right) = -\tfrac{1}{2}\eta_{ab}\,, \tag{3.20b}$$
$$\mathrm{tr}\left(K_a \tilde{K}_b\right) = 0\,. \tag{3.20c}$$

up to an overall factor of $\frac{l}{G}$ which can be absorbed in the definition of $\hat{k}$ by

$$\hat{k} = \frac{l}{4G}\,. \tag{3.21}$$

The result (3.20) explains the relative minus sign in the action (3.18): it arises because we simply replace A by $\tilde{A}$ in (3.19) without changing the bilinear form. Therefore, the Einstein-Hilbert action (3.12) of three-dimensional gravity for negative cosmological constant can be rewritten as the difference of two $\mathfrak{sl}(2,\mathbb{R})$-Chern-Simons theories.

Obviously, the Einstein-Hilbert action can only be rewritten as a Chern-Simons theory of the form (3.11) in three dimensions. However, as was mentioned before, one can still use the equations of motion (3.8) to describe maximally symmetric solutions in arbitrary dimensions. We will see that this observation is indeed useful in the context of four-dimensional Vasiliev theory.

3.1.2 *Oscillator Realization*

Using the definition $L_a = -\frac{1}{2}\epsilon_{abc}L^{bc}$ of the last section, one can rewrite the AdS_3-isometry algebra in a particularly convenient form

$$[P_a, P_b] = -\Lambda\,\epsilon_{abc}L^c\,, \tag{3.22a}$$
$$[L_a, P_b] = \epsilon_{abc}P^c\,, \tag{3.22b}$$
$$[L_a, L_b] = \epsilon_{abc}L^c\,. \tag{3.22c}$$

For reasons that will become apparent momentarily, we will rewrite these commutation relations in terms of $P_{\alpha\alpha} = \sigma^a_{\alpha\alpha}P_a$ and $L_{\alpha\alpha} = \sigma^a_{\alpha\alpha}L_a$ where $\sigma^a = (I, \sigma^1, \sigma^3)$ obeys

More generally any completely symmetric and traceless Lorentz tensor $T^{a(s)}$ is mapped to a completely symmetric multi-spinor $T^{\alpha(2s)} = (\sigma_a^{\alpha\alpha})^s T^{a(s)}$.

$$\sigma_a^{\alpha\alpha}\sigma_{b\,\alpha\alpha} = -2\eta_{ab}\,, \qquad \sigma_a^{\alpha\alpha}\sigma^a_{\beta\beta} = -\delta^\alpha_\beta\delta^\alpha_\beta\,. \tag{3.23}$$

One then obtains the following expressions

$$[P_{\alpha\alpha}, P_{\beta\beta}] = -\Lambda\,\epsilon_{\alpha\beta}L_{\alpha\beta}\,, \tag{3.24a}$$
$$[L_{\alpha\alpha}, P_{\beta\beta}] = \epsilon_{\alpha\beta}P_{\alpha\beta}\,, \tag{3.24b}$$
$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta}L_{\alpha\beta}\,. \tag{3.24c}$$

We will now discuss a particular realization of the isometry algebra (3.24) for negative cosmological constant. This realization will turn out to be crucial in constructing Vasiliev equations. To this end, we introduce a pair of commuting oscillators y^α with $\alpha = 0, 1$ obeying

$$y^\alpha y^\beta = y^\beta y^\alpha \,. \tag{3.25}$$

The spinorial indices can be lowered by using the antisymmetric epsilon tensor $\epsilon_{\alpha\beta}$ with $\epsilon_{01} = 1$ as follows

$$y_\alpha := y^\beta \epsilon_{\beta\alpha} \,. \tag{3.26}$$

For functions of the oscillators y_α, one then defines the associative *star product*:

$$(f \star g)(y) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2u \, \mathrm{d}^2v \; f(y+u) \, g(y+v) \exp\left(ivu\right) , \tag{3.27}$$

where we used the notation $v^\alpha u_\alpha =: vu = -uv$, which implies that contractions of identical commuting oscillators vanishes. Furthermore we will use the notation

$$\partial^y_\alpha := \frac{\partial}{\partial y^\alpha} \,, \qquad \partial^\alpha_y := \epsilon^{\alpha\beta} \partial^y_\beta \,, \tag{3.28}$$

from which we deduce the following relations

$$\partial^y_\alpha y^\beta = -\partial^\alpha_y y_\beta = \delta^\beta_\alpha , \tag{3.29a}$$
$$\partial^\alpha_y y^\beta = \epsilon^{\alpha\beta} \,, \tag{3.29b}$$
$$\partial^y_\alpha y_\beta = \epsilon_{\alpha\beta} \,. \tag{3.29c}$$

As a word of warning, we remark that relation (3.29a) implies that $\partial^\alpha_y = -\frac{\partial}{\partial y_\alpha}$.

From the definition (3.27), a differential version of the star product can also be obtained:

$$(f \star g)(y) = f(y) e^{-i \overleftarrow{\partial}_y \overrightarrow{\partial}_y} g(y) \,. \tag{3.30}$$

Therefore, the star product of these oscillators coincides with the Moyal product. Using the differential version of the star product, the following important identities can easily been derived

In $d = 3$, as we will see later, there are so called deformed oscillators for which the star is not the Moyal product. This is related to the fact that the higher-spin algebra is not unique in $d = 3$. On the contrary, for $d = 4$, the star product always coincides with the Moyal product and the higher-spin algebra is unique.

$$y_\alpha \star f(y) = (y_\alpha + i\partial^y_\alpha) f(y) \,, \tag{3.31a}$$
$$f(y) \star y_\alpha = (y_\alpha - i\partial^y_\alpha) f(y) \,. \tag{3.31b}$$

From which we conclude

$$[y_\alpha, f(y)]_\star = 2i \, \partial^y_\alpha f(y) \,, \tag{3.32a}$$
$$\frac{1}{2} \{y_\alpha, f(y)\}_\star = y_\alpha \, f(y) \,. \tag{3.32b}$$

We note that the last equation implies that the symmetric part of a string of star multiplied y_α-oscillators coincides with the ordinary product, e.g. $\frac{1}{2} y_\alpha \star y_\alpha = y_\alpha y_\alpha$. The first equation on the other hand implies that

$$\boxed{[y_\alpha, y_\beta]_\star = 2i\epsilon_{\alpha\beta}}\,. \tag{3.33}$$

Introducing this formalism is useful as one can easily show using (3.33) that the generators

$$L_{\alpha\alpha} = -\frac{i}{4} y_\alpha \star y_\alpha = -\frac{i}{2} y_\alpha y_\alpha \tag{3.34}$$

obey the commutation relations (3.24c) of $\mathfrak{sp}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R})$. From a more abstract point of view, the y_α-oscillators form an associative algebra with the multiplication rule given by the star product. We can turn this associative algebra into a Lie algebra by identifying the bracket with the star-commutator. We have shown that this Lie algebra has a subalgebra (spanned by the generators $L_{\alpha\alpha}$) which is isomorphic to $\mathfrak{sp}(2,\mathbb{R})$. One then usually says that the y_α-oscillators *realize* the $\mathfrak{sp}(2,\mathbb{R})$ algebra.

The generator $P_{\alpha\alpha}$ can also be realized by introducing an additional variable ϕ which has the following properties

$$\phi^2 = 1\,, \qquad\qquad y_\alpha\, \phi = \phi\, y_\alpha\,. \tag{3.35}$$

We then define

$$P_{\alpha\alpha} = \frac{\phi}{l} L_{\alpha\alpha}\,, \tag{3.36}$$

which obviously leads to (3.24) and a factor of the AdS radius l was introduced in order to balance the dimensions. We can then immediately write down the connection Ω

$$\Omega = \bar{\omega} + \bar{e} = \frac{1}{2}\bar{\omega}^{\alpha\beta} L_{\alpha\beta} + \frac{1}{2}\bar{e}^{\alpha\beta} P_{\alpha\beta}\,. \tag{3.37}$$

Note that (3.23) implies $\bar{e}^a P_a = -\frac{1}{2}\bar{e}^{\alpha\alpha} P_{\alpha\alpha}$ but we have absorbed the relative minus sign in the definition of Ω which leads to the following zero-curvature condition

$$\mathrm{d}\Omega - \Omega \wedge\star\Omega = 0\,. \tag{3.38}$$

As Vasiliev theory is formulated in terms of equations of motion, the particular choice of the bilinear form (3.20) which reproduces the Einstein-Hilbert action is irrelevant for its construction.

The introduction of the additional variable ϕ might seem a bit ad hoc but has a straightforward algebraical interpretation. Because of $\phi^2 = 1$, the projectors

$$\Pi_\pm = \frac{1}{2}(1 \pm \phi) \tag{3.39}$$

can be constructed and obey $\Pi_\pm^2 = \Pi_\pm$ and $\Pi_+\Pi_- = 0$. We then observe that $K_{\alpha\alpha} = K_a\sigma^a_{\alpha\alpha}$ and $\tilde{K}_{\alpha\alpha} = \tilde{K}_a\sigma^a_{\alpha\alpha}$ are given by

$$K_{\alpha\alpha} = \Pi_+ L_{\alpha\alpha}\,, \qquad \tilde{K}_{\alpha\alpha} = \Pi_- L_{\alpha\alpha}\,. \tag{3.40}$$

Therefore, the projectors $\Pi_\pm$ split the AdS$_3$-isometry algebra into

$$\mathfrak{so}(2,2) \cong \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})\,. \tag{3.41}$$

In summary, we have found a realization of the AdS$_3$ algebra in terms of y_α-oscillators and the variable ϕ. As will become clear in the following two sections, this realization is particularly convenient for the construction of the unfolded free equations.

3.1.3 *Free Higher-Spin Equations of Motion*

In this section, we will rewrite the free equations of motion for higher-spin gauge fields (2.49) using the y_α-oscillators. We will see that they take a particularly simple form in this language. To this end, we first recall that in three dimensions all tensors of the form $T^{a(s-1),b(t)}$ for $t > 1$ vanish. This implies that there are no extra fields in three dimensions and the only non-vanishing frame-like fields are

$$e^{a(s-1)} \qquad \text{and} \qquad \omega^{a(s-1),b}\,. \tag{3.42}$$

One can dualize the latter to obtain

$$\omega^{a(s-1)} := \epsilon^a{}_{bc}\,\omega^{a(s-2)b,c}\,. \tag{3.43}$$

As we mentioned in the last chapter, the frame-like equations of motion (2.53) in three dimensions are equivalent to

$$R^{a(s-1)} = 0\,, \tag{3.44a}$$
$$R^{a(s-1),b} = 0\,. \tag{3.44b}$$

For a Minkowski background, the curvatures are given by (2.45). We then rewrite (3.44) using the dualized connection $\omega^{a(s-1)}$ and the fact that all extra fields vanish

$$\nabla e^{a(s-1)} - \epsilon^a{}_{bc}\,\bar{e}^b \wedge \omega^{a(s-1)c} = 0\,, \tag{3.45a}$$
$$\nabla \omega^{a(s-1)} = 0\,. \tag{3.45b}$$

On an AdS$_3$ background, the free equations are more involved and we did not give closed expressions for the curvatures (2.49) in this case. In the following, we will construct these equations in closed form using spinorial notation. To this end, we map the local Lorentz indices of the generalized vielbein and spin-connection to spinorial indices

$$e^{a(s-1)} \longleftrightarrow e^{\alpha(2s-2)}\,, \qquad \omega^{a(s-1)} \longleftrightarrow \omega^{\alpha(2s-2)}\,. \tag{3.46}$$

It is useful to collect all the generalized spin-connections and vielbeins in the following one-form

$$\begin{aligned}\hat{\omega}(y,\phi|x) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{-i}{(2k)!}\left(\omega_{\alpha_1\ldots\alpha_{2k}}(x)+\frac{\phi}{l}\,e_{\alpha_1\ldots\alpha_{2k}}(x)\right)y^{\alpha_1}\ldots y^{\alpha_{2k}}\\ &=: \omega(y|x)+\phi\,e(y|x)\,. \end{aligned} \tag{3.47}$$

We will now make an ansatz for its equation of motion which we then show to coincide with the free equations (3.44) on an AdS$_3$ background. This ansatz is given by

$$D_\Omega\hat{\omega}(y,\phi|x) = 0\,, \tag{3.48}$$

where the AdS$_3$-covariant derivative for a differential form F of degree $|F|$ is given by

$$D_\Omega F := \mathrm{d}F - \Omega\wedge\star F + (-1)^{|F|}\,F\wedge\star\Omega\,. \tag{3.49}$$

The Lorentz covariant derivative ∇ only contains the spin-connection $\bar{\omega}$ of the AdS$_3$ background

$$\nabla F = \mathrm{d}F - \bar{\omega}\wedge\star F + (-1)^{|F|}F\wedge\star\bar{\omega}\,. \tag{3.50}$$

Since the background one-form Ω obeys a zero-curvature condition (3.38), the covariant derivative D_Ω is nilpotent

$$D_\Omega^2 = 0\,. \tag{3.51}$$

This shows that (3.48) is invariant under

$$\delta\hat{\omega}(y,\phi|x) = D_\Omega\xi(y,\phi|x)\,, \tag{3.52}$$

where $\xi(y,\phi|x)$ is some zero-form. Using the fact that

$$[L_{\alpha\alpha}, f(y)]_\star = y_\alpha\partial^y_\alpha f(y)\,, \tag{3.53}$$

which immediately follows from (3.32a), it is easy to check that (3.48) is in components equivalent to

$$\nabla e^{\alpha(2s-2)} - \bar{e}^{\alpha}{}_{\beta}\wedge\omega^{\beta\alpha(2s-3)} = 0\,, \tag{3.54a}$$

$$\nabla\omega^{\alpha(2s-2)} - \frac{1}{l^2}\,\bar{e}^{\alpha}{}_{\beta}\wedge e^{\beta\alpha(2s-3)} = 0\,. \tag{3.54b}$$

We will now show that these equations are indeed equivalent to the free equations of motion for the spin-s fields (3.44) on an AdS$_3$ background. As mentioned before, we did not give the generalized curvatures (2.49) for an AdS background in closed form. As we discussed however, the curvatures are determined by two properties: firstly, they should reduce to the curvatures of a Minkowski background upon taking the limit $\Lambda\to 0$ (or equivalently $l\to\infty$). Secondly, they should be gauge invariant also for non-vanishing cosmological constant Λ. From the fact that the gauge transformation (3.52) commutes with the y-number operator $N = y^\alpha \partial^y_\alpha$, it is clear that (3.54) is gauge invariant for each spin s. So we only need to show that the equations (3.54) indeed coincide with the free equations (3.45) for $l \to \infty$. From our discussion of the map between spinorial and local Lorentz indices, it is clear that

$$\nabla \omega^{\alpha(2s-2)} = 0 \qquad \leftrightarrow \qquad \nabla \omega^{a(s-1)} = 0\,, \tag{3.55}$$

and similarly for the generalized torsion constraint in spinorial (3.54a) and Lorentz notation (3.45a).

Therefore, the ansatz (3.48) indeed describes free gauge fields with spin $s = 1, 2, \ldots, \infty$ propagating on an AdS_3-background. Note that at the free level, it is merely convenient to combine all the spins in a single field $\hat{\omega}$ but any subset of higher-spin gauge fields would be consistent as well. However, as we will discuss, all higher-spins fields must generically be present at the non-linear level.

3.1.4 *Free Matter Equations of Motion*

Having constructed the free equations for the higher-spin gauge fields, we now turn our attention towards the matter sector. As we will see, Vasiliev theory contains a complex scalar field which is coupled to the higher-spin gauge fields. In principle, the free propagation of a scalar field on an AdS_3-background can straightforwardly be described by a Klein–Gordon equation. However in order to formulate the Vasiliev theory, one has to rewrite the Klein–Gordon equation in so-called *unfolded form*.

A set of equations of motion is said to be unfolded if it is of the form $d\Phi^i = F^i(\Phi)$ where Φ^i denotes various fields of possibly different form degrees. Note that in particular the equations of motion for the gauge sector (3.48) are in unfolded form.

In the following, we will simply state the correct form of the unfolded equation and then derive in detail its equivalence to a Klein–Gordon equation on an AdS_3-background. To this end, let us define the following zero-form

$$\hat{C}(y, \phi|x) = \sum_{s=0}^{\infty} \frac{1}{s!} \hat{C}_{\alpha_1 \ldots \alpha_s}(\phi|x)\, y^{\alpha_1} \ldots y^{\alpha_s}\,, \tag{3.56}$$

where we restrict to even s. The unfolded form of the Klein–Gordon equation is then given by

$$\nabla \hat{C} = \{\bar{e}, \hat{C}\}_\star\,. \tag{3.57}$$

This equation might look strange at first sight as it is a first order differential equation but we will show in the following that it is indeed completely equivalent to the Klein–Gordon equation. In order to do so, let us evaluate its right hand side using

$$\{L_{\alpha\beta}, f(y)\}_\star = -i \left(y_\alpha y_\beta - \partial^y_\alpha \partial^y_\beta \right) f(y)\,. \tag{3.58}$$

With the expansion (3.56) one then obtains

$$\begin{aligned}\{\bar{e}, \hat{C}\}_\star &= \frac{\phi}{2l}\,\bar{e}^{\alpha\beta}\,\{L_{\alpha\beta}, \hat{C}\}_\star \\ &= \sum_{s=0}^{\infty} \frac{\phi}{2i\,l}\frac{1}{s!}\Big(\bar{e}_{\beta_1\beta_2}\,\hat{C}_{\alpha_1\ldots\alpha_s}\,y^{\beta_1}y^{\beta_2}y^{\alpha_1}\ldots y^{\alpha_s} \\ &\qquad -s(s-1)\,\bar{e}^{\beta_1\beta_2}\,\hat{C}_{\alpha_1\ldots\alpha_s}\,\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}y^{\alpha_3}\ldots y^{\alpha_s}\Big)\,.\end{aligned} \tag{3.59}$$

By changing the summation index of the first sum from $s \to s-2$ and in the second sum from $s \to s+2$, we deduce that in components the equation of motion (3.57) is given by

$$\nabla \hat{C}_{\alpha(s)} = -i\,\frac{\phi}{l}\left(\bar{e}_{\alpha\alpha}\hat{C}_{\alpha(s-2)} - \frac{1}{2}\bar{e}^{\beta\beta}\hat{C}_{\beta\beta\alpha(s)}\right)\,, \tag{3.60}$$

where the Lorentz covariant derivative ∇ acts on components as

$$\nabla \hat{C}_{\alpha(s)} = \mathrm{d}\hat{C}_{\alpha(s)} + \bar{\omega}_\alpha{}^\beta \hat{C}_{\beta\alpha(s-1)}\,. \tag{3.61}$$

To analyze (3.60) in more detail, let us contract its spacetime index with a vielbein $\bar{e}^m_{\gamma\gamma}$ which leads to

$$\bar{e}^m_{\gamma\gamma}\nabla_m\hat{C}_{\alpha(s)} = -i\,\frac{\phi}{l}\left(\underbrace{\bar{e}^m_{\gamma\gamma}\bar{e}_{m\,\alpha\alpha}}_{\sim\,\epsilon_{\gamma\alpha}\epsilon_{\gamma\alpha}}\hat{C}_{\alpha(s-2)} - \frac{1}{2}\,\underbrace{\bar{e}^m_{\gamma\gamma}\bar{e}^{\beta\beta}_m}_{=-\frac{1}{4}\delta^\beta_\gamma\delta^\beta_\gamma}\,\hat{C}_{\beta\beta\alpha(s)}\right). \tag{3.62}$$

We will now consider the symmetric and antisymmetric part of the equation above with respect to exchange of γ and α indices. When restricted to the symmetric part, the first summand obviously vanishes and using $\phi^2 = 1$ one obtains

$$\hat{C}_{\alpha(s)} = 4i\phi\,l\,\bar{e}^m_{\alpha\alpha}\nabla_m\hat{C}_{\alpha(s-2)}\,. \tag{3.63}$$

Therefore, higher components of $\hat{C}(y)$ can be expressed as derivatives of lower components. Let us now project on the antisymmetric part with respect to $\gamma \leftrightarrow \alpha$ by multiplying (3.62) with $\epsilon^{\gamma\alpha}\epsilon^{\gamma\alpha}$. The second sum obviously vanishes and using the identity

$$\bar{e}^{\alpha\alpha}_m\bar{e}_{n\,\alpha\alpha} = -\frac{1}{2}g_{mn} \tag{3.64}$$

we obtain

$$\epsilon^{\gamma\alpha}\epsilon^{\gamma\alpha}\bar{e}^m_{\gamma\gamma}\nabla_m\hat{C}_{\alpha(s)} = \frac{3i}{2l}\,\phi\,\hat{C}_{\alpha(s-2)}\,. \tag{3.65}$$

Using this equation for $s = 2$ and the definition $\Phi := \hat{C}(y=0)$ combined with (3.63), we obtain

$$\epsilon^{\gamma\alpha}\epsilon^{\gamma\alpha}\,\bar{e}^m_{\gamma\gamma}\nabla_m\,\bar{e}^n_{\alpha\alpha}\nabla_n\Phi = \frac{3}{8l^2}\Phi \quad \Rightarrow \quad \boxed{\Box\Phi = -\frac{3}{4l^2}\Phi}\,. \tag{3.66}$$

We have therefore shown that the unfolded equations (3.57) contains a Klein–Gordon equation. The following comments are in order:

- Using the projectors $\Pi_\pm$ of (3.39), we can see that we actually obtained two Klein–Gordon equations for complex conjugated scalar fields. Decomposing $\Phi = \Pi_+\Phi + \Pi_-\Phi := \Phi_+ + \Phi_-$, we obtain

$$\Box\Phi_\pm = -\frac{3}{4l^2}\Phi_\pm \,. \tag{3.67}$$

The fact that Φ_+ and Φ_- are complex conjugates of each other originates from the fact that Vasiliev equations come with the reality condition $(\Phi_+)^\dagger = \Phi_-$ as we will discuss in Section 3.2.1.

- Let us note that the mass term $m^2 = -\frac{3}{4l^2}$ takes the value of a conformally coupled scalar on AdS$_3$.

- From (3.63) it is clear that all components of $\hat{C}(y,\phi|x)$ can be expressed in terms of derivatives of the scalar field Φ, i.e.

$$\hat{C}_{\alpha(2s)} = (4i\phi\, l\, \bar{e}^m_{\alpha\alpha}\nabla_m)^s\,\Phi \,. \tag{3.68}$$

- It can be shown that the equation (3.65) for $s > 2$ corresponds to derivatives of the Klein–Gordon equation. Therefore, the unfolded equation (3.57) is equivalent to a Klein–Gordon equation of a complex scalar. However, proving this is a bit tedious and can more elegantly be done using a cohomological analysis discussed in Appendix C.

For arbitrary backgrounds, the conformally coupled scalar obeys $\Box\Phi = \xi R\Phi$, where $\xi = \frac{d-2}{4(d-1)}$ is fixed by Weyl invariance of the action. Note also that $m^2 = -\frac{3}{4l^2}$ is above the Breitenlohner-Freedman bound, e.g. $m^2 \geq \frac{-d^2}{4l^2}$.

3.1.5 *Higher-Spin Algebra*

The algebra generated by even monomials in the y_α-oscillators

$$V_{\alpha(2s-2)} = \left(\frac{-i}{4}\right)^{s-1} y_{(\alpha_1}\star\cdots\star y_{\alpha_{2s-2})}\,, \tag{3.69}$$

with $s \geq 2$ realizes the *higher-spin algebra* $\mathfrak{hs}(\frac{1}{2})$. This algebra is defined by considering the associative algebra

$$\mathfrak{B}[\nu] = \frac{U(\mathfrak{sp}(2,\mathbb{R}))}{\langle C_2 + \frac{1}{4}(3-2\nu-\nu^2)\rangle}\,. \tag{3.70}$$

By $U(\mathfrak{sp}(2,\mathbb{R}))$, we denote the universal enveloping algebra of $\mathfrak{sp}(2,\mathbb{R})$. The denominator is given by the two-sided ideal generated by all elements taking a certain ν-dependent value for the quadratic Casimir C_2 of $\mathfrak{sp}(2,\mathbb{R})$. The associative algebra can be turned into a Lie algebra by identifying the commutator with the Lie bracket. By decomposing the resulting Lie algebra, one obtains the higher-spin algebra $\mathfrak{hs}(\lambda)$:

$$\mathfrak{B}[\nu] = \mathbb{R}\,\mathbb{1} \oplus \mathfrak{hs}(\lambda)\,, \tag{3.71}$$

where $\mathbb{1}$ denotes the unit element of the associative algebra $\mathfrak{B}[\nu]$ and the parameter λ is defined by $\lambda := \frac{1}{2}(\nu+1)$. We can easily determine the value of the quadratic Casimir for this oscillator realization:

$$C_2 \equiv -\frac{1}{2}L^{\alpha\beta}\star L_{\alpha\beta} = -\frac{1}{4}\{L^{\alpha\beta}, L_{\alpha\beta}\}_\star = -\frac{3}{4}\,, \tag{3.72}$$

which corresponds to $\nu = 0$ in (3.70). Therefore, the even monomials (3.69) indeed provide us with a realization of $\mathfrak{hs}(\lambda)$ for $\lambda = \frac{1}{2}$. We prove these statements in detail in Appendix B.2.

The reader might wonder at this stage if one can also construct oscillator realizations for other values of λ. This can indeed be done as we will discuss in the following. Let us stress however that we will mostly deal with the undeformed oscillators for the rest of this thesis. Deformed oscillators $\hat{y}_\alpha$ are defined which obey a deformed version of the commutation relations (3.33):

$$[\hat{y}_\alpha, \hat{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu k)\,, \tag{3.73}$$

where $\nu \in \mathbb{R}$ is the deformation parameter appearing in (3.70) as we will explain shortly. The *outer Kleinian* k obeys

$$k\hat{y}_\alpha = -\hat{y}_\alpha k\,, \qquad \phi k = k\phi\,, \qquad k^2 = 1\,. \tag{3.74}$$

Using the equation (3.73), one can check that

$$L_{\alpha\alpha} = -\frac{i}{4}\hat{y}_\alpha \star \hat{y}_\alpha \tag{3.75}$$

By considering both even and odd monomials in $\hat{y}_\alpha$ and keeping both k-parity sectors, the algebra $\mathfrak{shs}(\lambda)$ is realized. Its maximal finite subalgebra is the wedge algebra of the $\mathcal{N} = 2$ superconformal algebra. This is to be contrasted with $\mathfrak{sp}(2,\mathbb{R})$ for the case of $\mathfrak{hs}(\lambda)$ which forms the wedge algebra of the Virasoro algebra. See [42] for more details.

obeys the commutation relations (3.24c) of $\mathfrak{sp}(2,\mathbb{R})$. One then again considers even monomials in the deformed oscillators. For the deformed case, the quadratic Casimir is given by

$$C_2 = -\frac{1}{2}L^{\alpha\beta} \star L_{\alpha\beta} = -\frac{1}{4}(3 + 2\nu k - \nu^2)\,. \tag{3.76}$$

We then project on a sector of definite k-parity using the projectors

$$P_\pm = \frac{1}{2}(1 \pm k)\,. \tag{3.77}$$

This leads to a realization of $\mathfrak{hs}(\lambda)$ with $\lambda = \frac{1}{2}(1 \mp \nu)$ respectively. One could also introduce commuting oscillators and ensure the commutation relation (3.73) by modifying the star product (3.27). The resulting star product for $\nu \neq 0$ is no longer the Moyal product and its explicit expression is quite involved [43]. Recently, the structure constants of $\mathfrak{hs}(\lambda)$ were determined in [44] using the deformed oscillator representation.

3.1.6 *Twisted Fields*

Note that the one-form $\hat{\omega}$ obeys the equation $D_\Omega\hat{\omega} = 0$ and therefore transforms in the adjoint representation of the higher-spin algebra. However, the equations of motion (3.57) for $\hat{C}$ are not the analogous relation for a zero-form because the covariant derivative D_Ω acting on a zero-form F is given by

$$D_\Omega F = \nabla F - \frac{1}{2l}\,\bar{e}^{\alpha\beta}[\phi L_{\alpha\beta}, F]_\star\,, \tag{3.78}$$

as can be seen by (3.49). Therefore, the term proportional to the background vielbein comes with a commutator instead of an anticommutator. For this reason, the zero-form $\hat{C}$ is said to transform in the *twisted adjoint representation*. Note that the anticommutator in (3.57) is of great importance as it relates different components of the zero-form $\hat{C}$ contrary to the commutator. This can be seen by comparing (3.53) with (3.58). It was precisely this interplay of the different components that allowed us to show the equivalence of the Klein–Gordon equation and the unfolded equation of motion (3.57).

There is an elegant way to ensure that $\hat{C}$ transforms in the twisted adjoint representation. To this end, let us introduce an additional variable ψ obeying

$$\psi^2 = 1\,, \qquad \psi\phi = -\phi\psi\,, \qquad \psi y_\alpha = y_\alpha \psi\,. \tag{3.79}$$

Note that ψ and ϕ obey a Clifford algebra but we choose not to make this manifest in our notation for reasons that will become clear later. Using this definition, we can rewrite the equation of motion (3.57) for $\hat{C}$ compactly as

$$\begin{aligned} D_\Omega \left(\hat{C}\psi\right) &= \nabla(\hat{C}\psi) - \frac{1}{2l}\bar{e}^{\alpha\beta}\left[\phi L_{\alpha\beta}, \hat{C}\psi\right]_\star \\ &= \left(\nabla\hat{C} - \left\{\bar{e}, \hat{C}\right\}_\star\right)\psi = 0\,, \end{aligned} \tag{3.80}$$

where we have used the fact that the Lorentz derivative ∇ is ϕ independent and $[\phi f(y), g(y,\phi)\psi] = \{\phi f(y), g(y,\phi)\}\,\psi$ which follows immediately from $\psi\phi = -\phi\psi$.

For the interacting theory, it will turn out that we cannot restrict ourselves to the fields discussed so far. We will also need to consider so called *twisted fields* which transform in the twisted adjoint for one-forms and adjoint representation for zero-forms respectively.

The twisted zero-form $\tilde{C}$ obeys the following free equation of motion

$$D_\Omega \tilde{C} = 0\,. \tag{3.81}$$

Note that this equation of motion does not mix different components of $\tilde{C}$. Therefore, it is certainly not equivalent to a Klein–Gordon equation. Note also that by (3.52) this is precisely the form of a covariantly constant gauge parameter ξ. In a slight abuse of terminology, the components of twisted zero-form $\tilde{C}$ are therefore sometimes also referred to as Killing tensors.

On the other hand, the twisted one-form $\tilde{\omega}$ obeys the following equation

$$D_\Omega(\tilde{\omega}\psi) = 0\,. \tag{3.82}$$

Note that this relation does not decompose into independent equations for each component of $\tilde{\omega}$ and therefore does not describe a multiplet of higher-spin fields.

A spacetime interpretation for both the twisted zero and one-form is not known. Furthermore, their role within the Gaberdiel–Gopakumar duality is unclear. A crucial result of our analysis is that twisted fields can be set to zero up to second order in perturbations around an AdS_3-background as will be discussed. Given this observation, it is reasonable to expect that this will also be possible to higher orders and that one can therefore choose vanishing solutions for the twisted fields to arbitrary order in perturbation theory.

3.1.7 *Summary of Free Equations*

Over the last sections, we have outlined free unfolded equations of motion for the entire field content of three-dimensional Vasiliev theory. Upon inspecting these equations, we can see that it is convenient to rewrite them in a more compact form by combining the twisted and untwisted fields in a single field, i.e.

$$\boldsymbol{\omega}(y,\phi,\psi|x) = \hat{\omega}(y,\phi|x) + \tilde{\omega}(y,\phi|x)\psi\,, \tag{3.83a}$$
$$\mathbf{C}(y,\phi,\psi|x) = \hat{C}(y,\phi|x)\psi + \tilde{C}(y,\phi|x)\,. \tag{3.83b}$$

Such fields are therefore also functions of ψ. Note that the twisted fields of the one-form $\boldsymbol{\omega}$ is given by the ψ-dependent part whereas the zero-form $\mathbf{C}$ has the opposite decomposition. Using this definition, we can compactly summarize the free equations of motion

$$\boxed{D_\Omega \boldsymbol{\omega} = 0\,,} \tag{3.84a}$$
$$\boxed{D_\Omega \mathbf{C} = 0\,,} \tag{3.84b}$$

which are gauge invariant under

$$\boxed{\delta\boldsymbol{\omega} = D_\Omega \xi\,,} \tag{3.85a}$$
$$\boxed{\delta\mathbf{C} = 0\,,} \tag{3.85b}$$

where ξ is an arbitrary zero-form which may now also depend on ψ in addition to ϕ, y^α and x^m. One can also decompose these equations as follows

$$\boxed{D\hat{\omega} = 0\,,} \tag{3.86a}$$
$$\boxed{\tilde{D}\tilde{\omega} = 0\,,} \tag{3.86b}$$
$$\boxed{D\tilde{C} = 0\,,} \tag{3.86c}$$
$$\boxed{\tilde{D}\hat{C} = 0\,,} \tag{3.86d}$$

where we split the covariant derivative

$$D_\Omega\,\{\,g(y,\phi|x) + \tilde{g}(y,\phi|x)\,\psi\,\} = Dg(y,\phi|x) + \tilde{D}\tilde{g}(y,\phi|x)\,\psi\,, \tag{3.87}$$

using the adjoint and twisted-adjoint covariant derivatives defined by

$$D = \nabla - \frac{1}{2}\,\phi\,\bar{e}^{\alpha\alpha}[L_{\alpha\alpha}, \bullet]_\star = \nabla - \phi\,\bar{e}^{\alpha\alpha} y_\alpha \partial^y_\alpha\,, \tag{3.88}$$

$$\tilde{D} = \nabla - \frac{1}{2}\,\phi\,\bar{e}^{\alpha\alpha}\{L_{\alpha\alpha}, \bullet\}_\star = \nabla + \frac{i}{2}\,\phi\,\bar{e}^{\alpha\alpha}(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha)\,. \tag{3.89}$$

These equations will be of crucial importance for the fully non-linear equations of motion which we will discuss in the next section.

3.2 NON-LINEAR THEORY: VASILIEV EQUATIONS

In this section, we will introduce the non-linear *Vasiliev equations*. We will simply state Vasiliev equations in the next section and then show that they lead to the free unfolded equations of motion for higher-spin and scalar fields upon linearization. *From now on, in order to avoid convoluted notation, we will work in units in which the AdS radius is one, i.e.* $l = 1$.

3.2.1 *Masterfields and Vasiliev Equations*

Vasiliev theory is formulated in terms of masterfields $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{S}_\alpha$. The masterfields $\mathcal{W}$ and $\mathcal{B}$ contain the fields $\boldsymbol{\omega}$ and $\mathbf{C}$ respectively in addition to auxiliary fields. These auxiliary fields arise from introducing a set of oscillators z_α in addition to y_α. The star product for functions depending on both sets of oscillators is then given by

$$(f \star g)(y, z) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2u\, \mathrm{d}^2v\; f(y+u, z+u) \\ g(y+v, z-v)\exp\left(iv^\alpha u_\alpha\right), \tag{3.90}$$

which is consistent with our previous definition of the star product (3.27) for $f = f(y)$ and $g = g(y)$. The masterfields $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{S}_\alpha$ also depend on z_α-oscillators and contain the fields $\boldsymbol{\omega}$ and $\mathbf{C}$ as follows[2]

$$\mathcal{W}(y, z, \phi, \psi|x) = \boldsymbol{\omega}(y, \phi, \psi|x) + f(z, y, \phi, \psi|x)\,, \tag{3.91}$$

$$\mathcal{B}(y, z, \phi, \psi|x) = \mathbf{C}(y, \phi, \psi|x) + g(z, y, \phi, \psi|x)\,, \tag{3.92}$$

$$\mathcal{S}_\alpha(y, z, \phi, \psi|x) = f_\alpha(z, y, \phi, \psi|x)\,, \tag{3.93}$$

where all the functions f, g and f_α vanish for $z_\alpha = 0$. It is these functions that contain all auxiliary fields as we will see later. Note

2 More precisely, only the linear perturbations of $\boldsymbol{\omega}$ and $\mathbf{C}$ obey the free unfolded equations of motion. We will discuss this in detail in Section 3.2.3.

that this implies that the master field $\mathcal{S}_\alpha$ is purely auxiliary. Vasiliev equations are then given in terms of these masterfields and read

$$\boxed{\begin{aligned} \mathrm{d}\mathcal{W} &= \mathcal{W}\wedge\star\mathcal{W}\,, \\ \mathrm{d}\mathcal{B}\star\varkappa &= [\mathcal{W},\mathcal{B}\star\varkappa]_\star\,, \\ \mathrm{d}\mathcal{S}_\alpha &= [\mathcal{W},\mathcal{S}_\alpha]_\star\,, \\ 0 &= \{\mathcal{B}\star\varkappa,\mathcal{S}_\alpha\}_\star\,, \\ [\mathcal{S}_\alpha,\mathcal{S}_\beta]_\star &= -2i\epsilon_{\alpha\beta}(1+\mathcal{B}\star\varkappa)\,, \end{aligned}} \qquad \begin{aligned} &(3.94a)\\ &(3.94b)\\ &(3.94c)\\ &(3.94d)\\ &(3.94e)\end{aligned}$$

where $\varkappa := \exp iyz$ denotes the (inner) Kleinian which has the following properties

$$\varkappa\star\varkappa = 1\,, \qquad \varkappa\star f(y,z)\star\varkappa = f(-y,-z)\,. \tag{3.95}$$

Vasiliev equations are invariant under the following gauge transformations

$$\boxed{\begin{aligned} \delta\mathcal{W} &= \mathrm{d}\xi - [\mathcal{W},\xi]_\star\,, \\ \delta(\mathcal{B}\star\varkappa) &= [\xi,\mathcal{B}\star\varkappa]_\star\,, \\ \delta\mathcal{S}_\alpha &= [\xi,\mathcal{S}_\alpha]_\star\,, \end{aligned}} \qquad \begin{aligned} &(3.96a)\\ &(3.96b)\\ &(3.96c)\end{aligned}$$

where $\xi = \xi(y,z,\phi,\psi|x)$. Vasiliev equations are consistent partial differential equations with respect to the differential $\mathrm{d} = \partial_m \mathrm{d}x^m$. For reasons that will become apparent, we will refer to the first two Vasiliev equations (3.94a)-(3.94b) as the *dynamical Vasiliev equations* and the last three (3.94c)-(3.94e) as *non-dynamical Vasiliev equations*.[3]

In order to obtain equations of motions for only *bosonic* higher-spin and matter fields, one has to impose the following conditions[4]

Vasiliev equations can be formulated for deformed oscillators [2]. Not imposing the bosonic projection then leads to the presence of fermionic superpartners for the bosonic fields in the spectrum. This theory then turns out to be the higher-spin generalization of $\mathcal{N}=2$ supergravity [2].

$$\varkappa\star\mathcal{W}\star\varkappa = \mathcal{W}\,, \quad \varkappa\star\mathcal{B}\star\varkappa = \mathcal{B}\,, \quad \varkappa\star\mathcal{S}_\alpha\star\varkappa = -\mathcal{S}_\alpha\,, \tag{3.97}$$

which are also known as bosonic projections. Note that due to (3.95), this amounts to restricting to functions that are even in y_α, z_α for $\mathcal{W}$, $\mathcal{B}$ and odd for $\mathcal{S}_\alpha$. One also imposes the following (anti)hermiticity conditions [2],

$$\mathcal{W}^\dagger = -\mathcal{W}\,, \qquad \mathcal{S}_\alpha^\dagger = -\mathcal{S}_\alpha\,, \qquad \mathcal{B}^\dagger = \mathcal{B}\,, \tag{3.98}$$

where we defined hermitian conjugation of the oscillators by

$$(y_\alpha)^\dagger = y_\alpha\,, \qquad (z_\alpha)^\dagger = -z_\alpha\,, \quad \phi^\dagger = \phi\,, \qquad \psi^\dagger = \psi\,. \tag{3.99}$$

---

3 It is important to emphasize that the term "dynamical Vasiliev equations" is unrelated to the term "dynamical equations" introduced in Appendix C.

4 Note that this condition ensures that simultaneous expansion of the masterfields in both types of oscillators leads to expansion coefficients with only an even number of indices for $\mathcal{B}$ and $\mathcal{W}$. However, as we will discuss shortly, some of their expansion coefficients will be interpreted as the physical fields which are then ensured to be bosonic.

Using the property $(ab)^\dagger = b^\dagger a^\dagger$, it is easy to check that these conditions are compatible with Vasiliev equations. Upon recalling that $\mathbf{C} = \tilde{C} + \hat{C}\psi$, we see that $\mathcal{B}^\dagger = \mathcal{B}$ leads to

$$(\Phi_\pm)^\dagger = \Phi_\mp \,, \tag{3.100}$$

where $\Phi_\pm = \Pi_\pm \hat{C}(y=0)$. As discussed around (3.67), this implies that the theory contains one complex scalar field. Similarly, the reality condition $\mathcal{W}^\dagger = -\mathcal{W}$ ensures that the (higher-spin) vielbeins and spin-connections are real as can be seen by (3.47).

Note that at first sight the Vasiliev equation (3.94a) of the masterfield $\mathcal{W}$ might appear to lead to zero-curvature conditions for the gauge fields. We will see however that the z_α-dependence of this masterfield will lead to interactions between the scalar and gauge fields in this equation.

3.2.2 *AdS_3 Background*

We will now show that AdS_3 is an exact solution of Vasiliev equations (3.94). This is important as we will perturbatively expand around this solution to obtain the free unfolded equations (3.84). We make the following ansatz

$$\mathcal{W}^{(0)} = \Omega \,, \tag{3.101a}$$

$$\mathcal{B}^{(0)} = 0 \,, \tag{3.101b}$$

$$\mathcal{S}^{(0)}_\alpha = z_\alpha \,, \tag{3.101c}$$

where Ω is the AdS_3-connection given by (3.37). Inserting these expressions in the Vasiliev equations (3.94), we see that (3.94b) and (3.94d) are trivially satisfied. To check that (3.94e) and (3.94c) are satisfied, we use the following identity

$$[z_\alpha, f(y,z)]_\star = -2i\partial^z_\alpha f(y,z) \,, \tag{3.102}$$

which immediately shows that (3.94e) is satisfied. This identity also implies that (3.94c) is indeed satisfied, since $\mathrm{d}(z_\alpha) = \mathrm{d}x^n \partial_n (z_\alpha) = 0$ and Ω is z_α-independent. Lastly, the first equation (3.94a) yields

$$\mathrm{d}\Omega = \Omega \wedge \star \Omega \,, \tag{3.103}$$

which is precisely the zero-curvature condition (3.38) of the AdS_3-connection Ω. Therefore, the ansatz (3.101) is an exact solution of Vasiliev equations.

From this derivation, it is also clear that the ansatz (3.101) solves Vasiliev equations for any connection Ω, not only the one describing AdS_3, provided that Ω satisfies the flatness condition (3.103) and does not depend on the z_α-oscillators. This corresponds to solutions of Vasiliev theory which only contain gauge fields. They can then also be obtained from a Chern–Simons action [45]. By choosing a vacuum value of $\mathcal{B}^{(0)} = \nu$, one obtains a Chern–Simons theory with the gauge algebra $\mathfrak{hs}(\lambda)$ with $\lambda = \frac{1}{2}(1 \pm \nu)$. We will discuss this point in great detail shortly, but first we will show that linear fluctuations around the AdS_3 background lead to the correct free unfolded equations of motion.

3.2.3 *Linear Perturbations*

We will now expand in perturbations around the AdS_3 background found in the last section. To this end, it is advantageous to shift all the fields by their vacuum values:

$$\mathcal{S}_\alpha \to z_\alpha + 2i\mathcal{A}_\alpha\,, \qquad \mathcal{W} \to \Omega + \mathcal{W}\,, \qquad \mathcal{B} \to 2i\mathcal{B}\,. \tag{3.104}$$

Using the identity (3.102) and the bosonic projection (3.97), we can then rewrite the master equations (3.94) as

$$\begin{aligned}
D_\Omega \mathcal{W} &= \mathcal{W} \wedge \star \mathcal{W}\,, && (3.105a)\\
D_\Omega \mathcal{B} &= [\mathcal{W}, \mathcal{B}]_\star\,, && (3.105b)\\
\partial^z_\alpha \mathcal{W} &= D_\Omega \mathcal{A}_\alpha - [\mathcal{W}, \mathcal{A}_\alpha]_\star\,, && (3.105c)\\
\partial^z_\alpha \mathcal{B} &= [\mathcal{A}_\alpha, \mathcal{B}]_\star\,, && (3.105d)\\
\partial^z_\alpha \mathcal{A}^\alpha &= \mathcal{A}_\alpha \star \mathcal{A}^\alpha + \mathcal{B} \star \varkappa\,, && (3.105e)
\end{aligned}$$

where D_Ω denotes the AdS_3-covariant derivative defined in (3.49). The gauge transformations then become

$$\begin{aligned}
\delta \mathcal{W} &= D_\Omega \xi - [\mathcal{W}, \xi]_\star\,, && (3.106a)\\
\delta \mathcal{B} &= [\xi, \mathcal{B}]_\star\,, && (3.106b)\\
\delta \mathcal{A}_\alpha &= \partial^z_\alpha \xi + [\xi, \mathcal{A}_\alpha]_\star\,. && (3.106c)
\end{aligned}$$

We now expand the masterfields in perturbations around AdS_3 as follows

$$\mathcal{W} = \mathcal{W}^{(1)} + \mathcal{W}^{(2)} + \dots\,, \tag{3.107}$$

where $\mathcal{W}^{(i)}$ denotes the i-th order perturbation of $\mathcal{W}$ and we use analogous notation for $\mathcal{B}$ and $\mathcal{A}_\alpha$. Performing the shift (3.104) implies that $\mathcal{W}^{(0)} = 0$ and similarly for all other masterfields. Therefore, to linear order, Vasiliev equations (3.105) become

$$\begin{aligned}
D_\Omega \mathcal{W}^{(1)} &= 0\,, && (3.108a)\\
D_\Omega \mathcal{B}^{(1)} &= 0\,, && (3.108b)\\
\partial^z_\alpha \mathcal{W}^{(1)} &= D_\Omega \mathcal{A}^{(1)}_\alpha\,, && (3.108c)\\
\partial^z_\alpha \mathcal{B}^{(1)} &= 0\,, && (3.108d)\\
\partial^\alpha_z \mathcal{A}^{(1)}_\alpha &= -\mathcal{B}^{(1)} \star \varkappa\,. && (3.108e)
\end{aligned}$$

The non-dynamical equations (3.108c)-(3.108e) can be solved using the following identities[5]

$$\partial^z_\alpha f^\alpha(z,y) = g(z,y) \to f_\alpha = \partial^z_\alpha \epsilon(z,y) + z_\alpha \Gamma_1 \langle g(z,y)\rangle \,, \quad (3.110a)$$
$$\partial^z_\alpha f(z,y) = g_\alpha(z,y) \to f = \epsilon(y) + z^\alpha \Gamma_0 \langle g_\alpha(z,y)\rangle \,, \quad (3.110b)$$

where $\Gamma_n\langle\bullet\rangle$ stands for *homotopy integrals* defined as

$$\Gamma_n\langle f\rangle(z) := \int_0^1 \mathrm{d}t\, t^n\, f(tz)\,. \quad (3.111)$$

Therefore, the solutions for the non-dynamical equations are given by

$$\mathcal{B}^{(1)} = \mathbf{C}^{(1)}(y)\,, \quad (3.112a)$$
$$\mathcal{A}^{(1)}_\alpha = \partial^z_\alpha \epsilon^{(1)}(y,z) + z_\alpha \Gamma_1\langle \mathbf{C}^{(1)} \star \varkappa\rangle\,, \quad (3.112b)$$
$$\mathcal{W}^{(1)} = \boldsymbol{\omega}^{(1)}(y) + z^\alpha \Gamma_0 \langle D_\Omega \mathcal{A}^{(1)}_\alpha\rangle\,, \quad (3.112c)$$

where we have only made the dependence on the oscillators explicit. Note that these equations fully determine the z_α-dependence of the masterfields. This statement will also hold at higher orders in perturbation theory.

We impose the following gauge condition on the masterfield $\mathcal{A}_\alpha$:

$$z^\alpha \mathcal{A}_\alpha = 0\,, \quad (3.113)$$

which is usually referred to as *Schwinger–Fock gauge*. We will discuss this gauge choice in greater detail in Section 7.3. The Schwinger–Fock gauge implies that the homogeneous part $\partial^z_\alpha \epsilon^{(1)}(y,z)$ in (3.112b) vanishes. This is because $z^\alpha \partial^z_\alpha$ is the z_α-number operator which implies[6] that in the Schwinger–Fock gauge $\epsilon^{(1)}(y,z) = \epsilon^{(1)}(y)$ and therefore $\partial^z_\alpha \epsilon^{(1)}(y,z) = 0$. From this it also follows that the residual gauge freedom preserving the Schwinger–Fock gauge is given by z-independent gauge parameters $\xi^{(1)}(y)$ as can be seen by comparing with (3.106c) which gives

$$z^\alpha \delta \mathcal{A}^{(1)}_\alpha = z^\alpha \partial^z_\alpha \xi^{(1)}(z,y) \stackrel{!}{=} 0 \quad \Rightarrow \quad \xi^{(1)}(z,y) = \xi^{(1)}(y)\,. \quad (3.114)$$

5 For these equations to be valid the following compatibility condition has to be fulfilled

$$\partial^\alpha_z g_\alpha(z,y) = 0\,, \quad (3.109)$$

which however is guaranteed to hold for our purposes since Vasiliev equations are consistent.

6 Here we assume that $\epsilon^{(1)}(y,z)$ is analytic in the z_α-oscillators.

By evaluating the star products in (3.112) and decomposing them with respect to twisted and physical fields, one obtains after a straightforward calculation

$$\mathcal{W}^{(1)} = \hat{\omega}^{(1)}(y) + \tilde{\omega}^{(1)}(y)\psi + M_2\psi + \tilde{M}_2 + M_3\psi + \tilde{M}_3\,, \tag{3.115a}$$

$$\mathcal{B}^{(1)} = \hat{C}^{(1)}(y)\psi + \tilde{C}^{(1)}(y)\,, \tag{3.115b}$$

$$\begin{aligned}\mathcal{A}^{(1)}_\alpha = z_\alpha \int_0^1 \mathrm{d}t\, t\, \hat{C}^{(1)}(-zt)\, e^{ityz}\,\psi \\ + z_\alpha \int_0^1 \mathrm{d}t\, t\, \tilde{C}^{(1)}(-zt)\, e^{ityz}\,,\end{aligned} \tag{3.115c}$$

where we have defined

$$\begin{aligned} M_2 = \frac{\phi}{2}\bar{e}^{\alpha\alpha} \int_0^1 \mathrm{d}t(t-1)\, e^{ityz}\, z_\alpha \\ \left(y_\alpha(1-t) - i(1+t)t^{-1}\partial^z_\alpha\right)\hat{C}^{(1)}(-zt)\,,\end{aligned} \tag{3.116a}$$

$$\tilde{M}_2 = \phi\,\bar{e}^{\alpha\alpha} \int_0^1 \mathrm{d}t\,(1-t)t\, z_\alpha z_\alpha\, \tilde{C}^{(1)}(-zt)\, e^{ityz}\,, \tag{3.116b}$$

$$M_3 = \bar{\omega}^{\alpha\alpha} \int_0^1 \mathrm{d}t\,(1-t)t\, z_\alpha z_\alpha\, \hat{C}^{(1)}(-zt)\, e^{ityz}\,, \tag{3.116c}$$

$$\tilde{M}_3 = \bar{\omega}^{\alpha\alpha} \int_0^1 \mathrm{d}t\,(1-t)t\, z_\alpha z_\alpha\, \tilde{C}^{(1)}(-zt)\, e^{ityz}\,, \tag{3.116d}$$

and we have used the fact that $z^\alpha \mathrm{d}\mathcal{A}^{(1)}_\alpha = 0$ in the Schwinger–Fock gauge. We now need to insert these results in the dynamical Vasiliev equations (3.108a) and (3.108b) which leads to

$$D_\Omega\left(\hat{C}^{(1)}(y)\psi + \tilde{C}^{(1)}(y)\right) = 0\,, \tag{3.117a}$$

$$D_\Omega\left(\hat{\omega}^{(1)}(y) + \tilde{\omega}^{(1)}(y)\psi\right) = -D_\Omega\left(M_2\psi + \tilde{M}_2 + M_3\psi + \tilde{M}_3\right)\,. \tag{3.117b}$$

The left hand side of (3.117b) is z-independent and therefore also its right hand side has to have this property. This implies that we can evaluate it for $z = 0$ as all z-dependent terms have to cancel out anyways. Note however that the star product (3.90) does not commute with setting the z_α-oscillators to zero. Therefore, it is important that we first evaluate the star products contained in the covariant derivative D_Ω and only afterwards set all z-dependent factors to zero.

Using the explicit form of the star commutators (A.12a), (A.13a) and anti-commutator (A.13b) contained in the covariant derivatives, we obtain after some algebra

$$D\hat{\omega}^{(1)} = 0\,, \tag{3.118a}$$

$$\tilde{D}\tilde{\omega}^{(1)} = -\tfrac{\phi}{16}\, E^{\alpha\beta}\,(y_\alpha - i\partial^u_\alpha)(y_\beta - i\partial^u_\beta)\,\hat{C}(u)\Big|_{u=0} \tag{3.118b}$$

$$D\tilde{C}^{(1)} = 0\,, \tag{3.118c}$$

$$\tilde{D}\hat{C}^{(1)} = 0\,, \tag{3.118d}$$

where we have defined

$$E^{\alpha\alpha} = \bar{e}^{\alpha}{}_{\beta} \wedge \bar{e}^{\alpha\beta} \,. \tag{3.119}$$

However, these are not quite the free unfolded equations (3.86) as there is a source term in the equation of motion for the twisted zero-form $\tilde{\omega}$. It was shown in [37] that one can remove this source term by performing a field redefinition $\tilde{\omega}^{(1)} \to \tilde{\omega}^{(1)} + M_1'\psi$ with

$$M_1' = \frac{\phi}{4}\, \bar{e}^{\alpha\beta} \int_0^1 \mathrm{d}t\, (t^2-1)\, (y_\alpha - it^{-1}\partial^y_\alpha)(y_\beta - it^{-1}\partial^y_\beta)\, \hat{C}^{(1)}(ty)\,. \tag{3.120}$$

However, we observed in [26] that the contribution

$$R := \frac{\phi}{4}\, \bar{e}^{\alpha\alpha} \int_0^1 \mathrm{d}t\, (t^2-1) \left(y_\alpha y_\alpha - t^{-2}\partial^y_\alpha \partial^y_\alpha\right) \hat{C}^{(1)}(ty)\,. \tag{3.121}$$

is annihilated by the twisted covariant derivative, i.e.

$$\tilde{D}R = 0\,. \tag{3.122}$$

We derive this fact in Appendix B.3 using a different proof than the one given in [26]. Therefore, we can add R freely to (3.120) without changing the effect of the redefinition. This shows that there is a one parameter ambiguity in this field redefinition and we therefore define

$$M_1 := \frac{\phi}{4} \bar{e}^{\alpha\alpha} \int_0^1 \mathrm{d}t\, (t^2-1) \left(g_0\, y_\alpha y_\alpha + 2iy_\alpha t^{-1}\partial^y_\alpha - g_0\, t^{-2}\partial^y_\alpha \partial^y_\alpha\right) \hat{C}^{(1)}(ty)\,. \tag{3.123}$$

The free parameter g_0 will have important implications as we will see later. This field redefinition is local in the sense that it contains only a finite number of derivatives for fixed spin, as can be seen by comparing with (3.68). After performing this field redefinition, one therefore obtains the free unfolded equations (3.86). We have thus shown that Vasiliev equations provide a non-linear theory of higher-spin gauge fields coupled to a complex scalar field.

3.3 OUTLOOK

As we have seen in this chapter, Vasiliev theory provides us with a consistent non-linear theory of higher-spin gauge and scalar fields (and an additional twisted sector). However, this theory is formulated in a non-standard way in terms of Vasiliev equations for masterfields.

Obtaining equations of motion expressed in terms of physical fields only is a highly non-trivial task. This is because one has to first solve for the z-dependence of the masterfields using the non-dynamical Vasiliev equations (3.105c)-(3.105e). Then one has to insert these solutions in the dynamical equations (3.105a)-(3.105b) and evaluate the star products in the resulting expressions. By this procedure, equations of motion in terms of physical and twisted fields are obtained. One then has to ensure that the twisted fields can be set to zero consistently - possibly by performing field redefinitions - in order to arrive at equations for the physical fields only.

In the next chapter, we will consider the theory for vanishing zero-form $\mathcal{B}$ (and twisted fields). As we discussed in Section 3.2.2, this case leads to a particularly simple description of the theory which we will study in metric-like language to obtain a geometrical intuition of the theory.

In Chapter 5, we will then generalize the discussion of this chapter to the second order for which interactions become relevant.

4

METRIC-LIKE THEORY

In this chapter, we will consider a truncation of three-dimensional Vasiliev theory for vanishing zero-form. This truncation contains gauge fields with spin $s = 2 \dots N$ only. The dynamics of these gauge fields can be described by a simple generalization of the Einstein–Hilbert action in the frame-like formalism. The frame-like theory being so simple it is then tempting to reformulate this theory in terms of metric-like variables. This could allow for a more geometrical interpretation of the theory and might lead to a better understanding of the non-linear Vasiliev theory.

For example in [46], corrections to the entropy of black holes with non-vanishing higher-spin charges were calculated using the metric-like approach. The result did not agree with the one derived from thermodynamic considerations in the frame-like theory in [47, 48]. This discrepancy was later resolved in [49] by a careful Hamiltonian analysis and vindicated the metric-like results. In many ways our work can be seen as a follow-up to [46]. This chapter is structured as follows:

- In Section 4.1, we will first show that for vanishing zero-form $\mathcal{B}$ the tower of higher-spin fields of three-dimensional Vasiliev theory can be truncated to a finite subset.
- In Section 4.2, we will then discuss how the fields of the truncated theory can be rewritten in terms of metric-like variables.
- Section 4.3 outlines an algorithm which allows one to translate frame-like quantities to their metric-like counterparts.
- In Section 4.4, this algorithm is used to determine the metric-like gauge transformations up to cubic order in the spin-3 field.
- Section 4.5 discusses the gauge algebra of the metric-like theory.

4.1 TRUNCATION OF HIGHER-SPIN ALGEBRA

As we discussed in Section 3.2.2, any connection $\boldsymbol{\omega}(y, \phi|x)$ which is z_α-independent and fulfills

$$\mathrm{d}\boldsymbol{\omega} - \boldsymbol{\omega} \wedge \star \boldsymbol{\omega} = 0\,, \tag{4.1}$$

forms an exact solution of Vasiliev equations (3.94) for vanishing zero-form $\mathcal{B}$. We also only consider the case of vanishing twisted one-form.

By defining $\boldsymbol{\omega}(y, \phi|x) = \Pi_+ A(y|x) + \Pi_- \tilde{A}(y|x)$, with the projectors $\Pi_\pm$ given by (3.39), the equation of motion decompose into

$$\mathrm{d}A - A \wedge \star A = 0\,, \qquad \mathrm{d}\tilde{A} - \tilde{A} \wedge \star \tilde{A} = 0\,, \tag{4.2}$$

and can therefore be derived from the action[1]

$$S = S_{CS}[A] - S_{CS}[\tilde{A}]\,. \tag{4.3}$$

The Chern–Simons action is given by

$$S_{CS}[A] = \frac{\hat{k}}{4\pi}\int \mathrm{tr}\left(A \wedge \star \mathrm{d}A - \frac{2}{3} A \wedge \star A \wedge \star A\right)\,. \tag{4.4}$$

We note that while the derivation of (4.1) in Section 3.2.2 was for undeformed oscillators only, one can generalize this argument also for the deformed case [2] by choosing the vacuum value $\mathcal{B}^{(0)} = \nu$ for the zero-form. The connection $\boldsymbol{\omega}$ then also obeys (4.1) but is an even polynomial in the deformed oscillators (3.73) and also depends on the outer Kleinian k. One then defines the trace in (4.4) by

$$\mathrm{tr}(k) = -\nu\,, \qquad\qquad \mathrm{tr}(1) = 1\,, \tag{4.5}$$

and vanishing trace for all monomials of the deformed oscillators $\hat{y}$ [50]. As discussed in Section 3.1.5, the higher-spin algebra $\mathfrak{hs}(\lambda)$ for $\lambda = \frac{1}{2}(1+\nu)$ is realized by the generators

$$V_{\alpha(2s)} = \mathcal{N}_s\, P_-\, \hat{y}_{(\alpha_1} \star \cdots \star \hat{y}_{\alpha_{2s})}\,, \qquad\qquad s > 0\,, \tag{4.6}$$

with normalization $\mathcal{N}_s = (-\frac{i}{4})^s$ and $P_- = \frac{1}{2}(1-k)$. One can then check that the trace introduced above gives for the generators

A readable derivation of this result can be found in the appendix of [50]. Another seminal paper on the subject is [51].

$$\mathrm{tr}\left(V_{\alpha(2s)} \star V_{\beta(2m)}\right) = \frac{\mathcal{N}_s^2}{(2s)!}\,\mathcal{K}(2s)\,\delta_{s,m}\,\epsilon_{\alpha_1\beta}\ldots\epsilon_{\alpha_{2s}\beta} \tag{4.7}$$

where we defined

$$\mathcal{K}(2s) = (-1)^s\left(1 + \frac{\nu}{2s+1}\right)\prod_{l=0}^{s-1}\left(1 - \frac{\nu^2}{(2l+1)^2}\right)\,. \tag{4.8}$$

We first consider the case $\nu \geq 0$. By (4.8), the Killing form only degenerates for the choice $\nu = 2p+1$ with $p \in \mathbb{N}$, which corresponds to $\lambda = p+1$. Indeed, we see from (4.7) that the generators $V_{\alpha(2s)}$ with $s \geq p+1$ lead to

$$\mathrm{tr}\left(V_{\alpha(2s)} \star X\right) = 0\,, \tag{4.9}$$

for all $X \in \mathfrak{hs}(p+1)$. The vector space I_A spanned by all these generators $V_{\alpha(2s)}$ forms a two-sided ideal of the corresponding associative algebra $\mathfrak{B}(2p+1)$, because of $\mathrm{tr}\,((I_A \star X) \star Y) = \mathrm{tr}\,(I_A \star (X \star Y)) = 0$ and $\mathrm{tr}\,((X \star I_A) \star Y) = \mathrm{tr}\,(I_A \star (Y \star X)) = 0$ for arbitrary $X, Y \in \mathfrak{B}(2p+1)$. This in turn induces an ideal I for the Lie algebra $\mathfrak{hs}(p+1)$. As a result, we conclude that only the quotient algebra $\mathfrak{hs}(p+1)/I$,

1 The relative minus sign between the two Chern–Simons actions can be fixed by requiring that the action agrees with the Einstein–Hilbert action upon considering the truncation to spin-2 gauge fields only.

which consists of all equivalence classes $x + I$ for $x \in \mathfrak{hs}(p+1)$, will contribute to the Chern–Simons action. This quotient algebra has dimension

$$d = \sum_{l=1}^{p}(2l+1) = (p+1)^2 - 1\,. \tag{4.10}$$

This indicates that this quotient algebra is isomorphic to $\mathfrak{sl}(p+1|\mathbb{R})$, i.e.

$$\frac{\mathfrak{hs}(p+1)}{I} \cong \mathfrak{sl}(p+1|\mathbb{R})\,. \tag{4.11}$$

Similar conclusions can be drawn for $\nu < 0$.[2] However, we have only shown that the quotient algebra is semi-simple and therefore we have not established this isomorphism rigorously. We give a complete proof in Appendix B.2.3.

4.2 METRIC-LIKE FIELDS

As we have seen in the last section, for the special value of $\lambda = N$ with $N \in \mathbb{N}$, the fields A and $\tilde{A}$ can be chosen to take value in $\mathfrak{sl}(N|\mathbb{R})$. In order to study the dynamics of Vasiliev theory beyond the linear order, we will therefore consider the case $N = 3$ which corresponds to truncating the infinite tower of higher-spin fields to a theory with spin-2 and spin-3 gauge fields only.

In this case the generalized vielbein and spin-connection are given by

$$e = e_m^{\mathcal{A}}\, J_{\mathcal{A}}\, \mathrm{d}x^m\,, \qquad\qquad \omega = \omega_m^{\mathcal{A}}\, J_{\mathcal{A}}\, \mathrm{d}x^m\,, \tag{4.12}$$

where $J_{\mathcal{A}}$ form a basis of $\mathfrak{sl}(3|\mathbb{R})$,

$$[J_{\mathcal{A}}, J_{\mathcal{B}}] = f_{\mathcal{ABC}}\, J^{\mathcal{C}}\,. \tag{4.13}$$

Similar to our discussion for three-dimensional gravity in Section 3.1.1, the action (4.4) (for fields now taking values in the factor algebra $\mathfrak{sl}(3|\mathbb{R})$ instead of the full higher-spin algebra) can be rewritten as

$$S = \frac{1}{16\pi G}\int \mathrm{tr}\left(e\wedge R + \frac{1}{3l^2} e\wedge e\wedge e\right)\,, \tag{4.14}$$

where

$$R = \mathrm{d}\omega + \omega\wedge\omega \quad \Leftrightarrow \quad R^{\mathcal{A}} = \mathrm{d}\omega^{\mathcal{A}} + \frac{1}{2} f^{\mathcal{A}}{}_{\mathcal{BC}}\, \omega^{\mathcal{B}}\wedge\omega^{\mathcal{C}}\,, \tag{4.15}$$

2 This case corresponds to $\lambda = \frac{1}{2}(1-|\nu|)$. The bilinear form only degenerates for $|\nu| = 2p+1$. All generators with $s \geq p$ lead to vanishing trace. Therefore, the quotient algebra has dimension $p^2 - 1$, which indicates that $\mathfrak{hs}(p)/I \cong \mathfrak{sl}(p|\mathbb{R})$.
An analogous analysis can be performed for the generators (4.6) with P_+ instead of P_- as this corresponds to $\nu \to -\nu$. Also note that there is an unfortunate typo in the original reference [50]: upon discussing the ideals for $P_\pm$ the role of the projectors should be swapped.

is the generalized curvature and tr denotes the trace in the fundamental representation of $\mathfrak{sl}(3|\mathbb{R})$. This action is invariant under generalized local Lorentz rotations

$$\delta_\Lambda e_n^{\mathcal{A}} = f^{\mathcal{A}}{}_{\mathcal{BC}}\, \Lambda^{\mathcal{B}} e_n^{\mathcal{C}}\,, \tag{4.16a}$$
$$\delta_\Lambda \omega_n^{\mathcal{A}} = D_n \Lambda^{\mathcal{A}}\,, \tag{4.16b}$$

and generalized local translations

$$\delta_\Xi e_n^{\mathcal{A}} = D_n \Xi^{\mathcal{A}}\,, \tag{4.17a}$$
$$\delta_\Xi \omega_n^{\mathcal{A}} = \frac{1}{l^2}\, f^{\mathcal{A}}{}_{\mathcal{BC}}\, e_n^{\mathcal{B}}\, \Xi^{\mathcal{C}}\,, \tag{4.17b}$$

where the covariant derivative D_n is defined as

$$D_m v^{\mathcal{A}} := \partial_m v^{\mathcal{A}} + f^{\mathcal{A}}{}_{\mathcal{BC}}\, \omega_m^{\mathcal{B}}\, v^{\mathcal{C}}\,. \tag{4.18}$$

Having obtained such a simple frame-like theory, we now want to map this theory into metric-like variables. In order to do so, one has to first express the generalized spin-connection in terms of the generalized vielbein by its equation of motion

$$D_{[m} e_{n]}^{\mathcal{A}} = 0\,, \tag{4.19}$$

where D_m now includes both the generalized spin-connection and the Christoffel symbols

$$D_m e_n^{\mathcal{A}} = \partial_m e_n^{\mathcal{A}} + f^{\mathcal{A}}{}_{\mathcal{BC}}\, \omega_m^{\mathcal{B}}\, e_n^{\mathcal{C}} - \Gamma^p{}_{mn}\, e_p^{\mathcal{A}}\,. \tag{4.20}$$

Then all expression have to be rewritten in terms of the metric and the spin-3 fields by using

$$g_{nm} = \kappa_{\mathcal{AB}}\, e_n^{\mathcal{A}}\, e_m^{\mathcal{B}}\,, \tag{4.21a}$$
$$\phi_{nmr} = \frac{1}{3!} d_{\mathcal{ABC}}\, e_n^{\mathcal{A}} e_m^{\mathcal{B}} e_r^{\mathcal{C}}\,. \tag{4.21b}$$

The definitions for the Killing form $\kappa_{\mathcal{AB}}$ and the symmetric structure constant $d_{\mathcal{ABC}}$ can be found in Appendix A. The identification above was first proposed in [15]. It is motivated by the fact that the chosen frame-like expressions are the only generalized local Lorentz invariant polynomials of vielbeins $e_n^{\mathcal{A}}$ which are completely symmetric in two or three spacetime indices respectively.

Note that the generalized vielbein $e_m^{\mathcal{A}}$ is not invertible as it is not a square matrix. This fact and the additional complication that vielbeins can now combine in two structures (4.21) makes the translation between frame-like and metric-like quantities quite involved. In the next section, we will present an algorithm which allows one to systematically translate frame-like to metric-like quantities in a perturbative expansion in the spin-3 field.

4.3 FROM FRAME- TO METRIC-LIKE: AN ALGORITHM

In the following, we will present an algorithm to translate from frame- to metric-like quantities. We will first introduce this algorithm for quantities which do not involve covariant derivatives. We will then generalize the algorithm such that it also works if covariant derivatives are present. We will also discuss why this algorithm leads to a unique translation despite the appearance of free parameters. Our presentation will be relatively concise and more details including an explicit example can be found in our original publication [25].

4.3.1 *Algorithm without Derivatives*

In this section, we will present the algorithm which translates a frame-like quantity in its metric-like counterpart.

It is based on a perturbative expansion in the spin-3 field. For this purpose, it is useful to split the generators of $\mathfrak{sl}(3,\mathbb{R})$ in generators J_a of the principally embedded $\mathfrak{sl}(2,\mathbb{R})$, which we label by lower-case Latin indices, and the remaining generators J_A which are orthogonal to the J_a with respect to the Killing form of $\mathfrak{sl}(3,\mathbb{R})$ and labeled by capital Latin indices. This leads to the following decomposition of the generalized vielbein

$$e^{\mathcal{A}} = \left(e^a, E^B\right) . \tag{4.22}$$

Using this decomposition, we can expand the metric-like fields

$$\phi_{nnn} = \tfrac{1}{6}\, d_{Abc}\, E^A_n e^b_n e^c_n + \tfrac{1}{6}\, d_{ABC}\, E^A_n E^B_n E^C_n \,, \tag{4.23a}$$

$$g_{nn} = \kappa_{ab}\, e^a_n e^b_n + \kappa_{AB}\, E^A_n E^B_n =: \bar{g}_{nn} + g^{(2)}_{nn} \,. \tag{4.23b}$$

The algorithm then works as follows: consider a frame-like quantity which does not contain any covariant derivative and has spacetime indices $n_1 \ldots n_m$ and all frame-like indices contracted. For its metric-like counterpart, we first make an ansatz involving all possible contractions of metric-like fields up to a certain order n in the spin-3 field ϕ_{mmm}. The algorithm then proceeds in five steps:

1. Use (4.21) to express the metric-like ansatz in terms of frame-like variables. Expand both the metric-like ansatz and the frame-like quantity in terms of E^A up to order n and subtract them from each other.

2. Consider each order from 0 to n independently. For each of them one obtains

$$\sum_i c^{(i)}\, t^{(i)}_{a(p_i)A(l_i)}\, K^i(\{e,E\})^{a(p_i)A(l_i)}_{n(s)} = 0 \,. \tag{4.24}$$

   The $t^{(i)}_{a(p_i)A(l_i)}$ are $\mathfrak{sl}(2,\mathbb{R})$-invariant tensors involving the symmetric structure constant and Killing form. Furthermore $K^i(\{e,E\})$

are tensors obtained by contractions of vielbeins. Last but not least, the $c^{(i)}$ are the free coefficients of the metric-like ansatz.[3]

3. Replace the E^A by

$$E^A_m \to E^A_n \delta^n_m\,, \tag{4.25}$$

where one has to use the following expression for the Kronecker symbol

$$\delta^n_m = \bar{g}^{np}\, e^a_p e^b_n\, \kappa_{ab}\,. \tag{4.26}$$

4. Impose

$$\bar{g}^{nm}\, e^a_m e^b_n = \kappa^{ab}\,, \tag{4.27}$$

for all contractions of this type. After this replacement all tensors K^i in the sum (4.24) will be of the same form and we obtain

$$\left(\sum_i c^{(i)}\, \tilde{t}^{(i)}_{a(q)A(k)}\right) K(\{e,E\})^{a(q)A(k)}_{n(s)} = 0\,. \tag{4.28}$$

5. One can solve (4.28) for the coefficients $c^{(i)}$ by stripping off the tensor K

$$\mathcal{P}\sum_i c^{(i)}\, \tilde{t}^{(i)}_{a(q)A(k)} = 0\,, \tag{4.29}$$

where $\mathcal{P}$ denotes a projector which imposes the symmetries of K.

We expand to order n in the first step because the spin-3 field depends linearly on E^A as can be seen by comparing with (4.23). Therefore, we only have to consider terms up to order n in E^A because higher order terms would receive contributions from metric-like expressions with more than n spin-3 fields. The replacement (4.25) and imposing (4.27) guarantees that all K^i will be of the same form and all free indices are carried by $\mathfrak{sl}(2,\mathbb{R})$-vielbeins. A basic example for the application of this algorithm can be found in Section 3.3 of [25].

4.3.2 *Algorithm with Derivatives*

In the last section, we saw that the algorithm outlined there crucially relies on the fact that the tensors K^i of (4.24) can be brought in the same form K to obtain (4.28). Therefore, for expressions involving covariant derivatives, an additional complication arises: the covariant derivative

3 The tensors $K^i(\{e,E\})$ can also depend on the epsilon tensor and we assume that all terms in the sum contain the same number of them with upper spacetime indices only. Furthermore, some of the $c^{(i)}$ are equal to 1 if they originate from the frame-like quantity which is to be translated.

can either act on an $\mathfrak{sl}(2,\mathbb{R})$-vielbein e^a or a vielbein E^A. For our algorithm to work, we need to be able to express $D_m e^a_n$ in terms of $D_m E^A_n$ such that we can hope to obtain an overall tensor structure K as in (4.28) which we can then strip off. This can indeed be achieved by using covariant constancy of the metric, $\nabla_p g_{nm} = 0$, to obtain

$$\kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_m\, D_p e^{\mathcal{B}}_n = 0\,. \tag{4.30}$$

By expansion this leads to

$$D_m e^c_n = -\kappa_{AB}\, \bar{g}^{ro}\, e^c_o E^A_r D_m E^B_n\,. \tag{4.31}$$

We can then generalize the algorithm by modifying step 1 and 3 as follows:

1'. Proceed as previously discussed but also replace all covariant derivatives acting on $\mathfrak{sl}(2,\mathbb{R})$-vielbeins by (4.31).

3'. Proceed as previously discussed but also replace covariant derivatives of E^A by

$$D_n E^A_m \to \delta^p_n\, \delta^o_m\, D_p E^A_o\,, \tag{4.32}$$

where one has to use expression (4.26) for the Kronecker symbol.

All other steps are unchanged.

4.3.3 *Dimensional Dependent Identities*

We encountered various examples for which the solution for the $c^{(i)}$ in (4.28) is not unique and contains free parameters. However, this does not imply that the frame-like quantity has a family of metric-like counterparts. These free parameters are due to the existence of dimensional dependent identities (DDIs). An example for such an DDI is given by

$$\delta^m_{[p}\delta^n_o\delta^u_r\delta^t_{q]} = 0\,, \tag{4.33}$$

which is identically zero in three dimension. It turns out that all DDIs can be obtained in a similar fashion by over-symmetrization. A systematic way of obtaining all DDIs for a given set of metric-like fields is described in [52]. Imposing these DDIs we always found unique metric-like expressions.

4.4 GAUGE TRANSFORMATIONS

In this section, we will use the algorithm discussed in the last section to determine the spin-3 transformations of the metric-like fields to cubic order from the frame-like theory.

The frame-like gauge transformations are generalized local Lorentz transformations (4.16a) and generalized local translations (4.17). Consider the metric-like fields ϕ_{nnn} and g_{nn}. A spin-2 gauge transformation parameterized by a vector field ξ^m acts on these fields as follows

$$\delta^{(2)}_\xi \phi_{nnn} = \mathcal{L}_\xi \phi_{nnn} := \xi^r \nabla_r \phi_{nnn} + \phi_{nnr} \nabla_n \xi^r \,, \tag{4.34}$$

$$\delta^{(2)}_\xi g_{nn} = \mathcal{L}_\xi g_{nn} := \xi^r \nabla_r g_{nn} + g_{nr} \nabla_n \xi^r \,, \tag{4.35}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the vector field ξ^m. For the spin-3 gauge transformations, the transformation behavior of the metric-like fields is only known for the free theory and is then given by (2.6) for the general case of spin-s. This implies that the spin-3 transformation is given by

$$\delta^{(3)}_\xi \phi_{nnn} = \frac{1}{3} \nabla_n \left(\xi_{nn} - \frac{1}{3} g_{nn}\, \xi_m{}^m \right) + \dots \tag{4.36}$$

$$\delta^{(3)}_\xi g_{nn} = 0 + \dots \,, \tag{4.37}$$

where the ellipses denote higher order terms containing spin-3 fields. Furthermore, we explicitly project on the traceless part of the spin-2 gauge parameter ξ^{nn} for reasons that will become apparent.

It is natural to combine spin-2 and spin-3 transformations into a single parameter $\xi := (\xi^n, \xi^{nn})$. As we want to relate frame-like with metric-like gauge transformations, we need to construct a map

$$\xi = (\xi^n, \xi^{nn}) \mapsto \Xi(\xi) \,, \tag{4.38}$$

such that

$$\delta_{\Xi(\xi)} \phi_{nnn} = \delta^{(2)}_\xi \phi_{nnn} + \delta^{(3)}_\xi \phi_{nnn} \,, \tag{4.39}$$

and similarly for the metric. This map is not unique even if we have fixed the expressions of the metric-like fields in terms of the frame-like fields such that no field redefinitions are possible: we can still redefine the spin-3 gauge parameter ξ^{nn} by terms at least linear in the spin-3 field ϕ_{nnn} such that the linearized gauge transformations are unchanged. In the following section, a particularly natural map will be constructed which is valid to all orders in the spin-3 field. We will then use the algorithm outlined before to determine the spin-3 transformations of the metric-like fields.

4.4.1 *A Natural Map*

The map $\xi \mapsto \Xi(\xi)$ is linear and can therefore be written as

$$\Xi^{\mathcal{A}}(\xi) = S^{\mathcal{A}}{}_n \, \xi^n + S^{\mathcal{A}}{}_{nn} \, \xi^{nn} \,, \tag{4.40}$$

with possibly field dependent matrices S. Pure spin-2 gauge transformations are given by

$$S^{\mathcal{A}}{}_n = e^{\mathcal{A}}_n \,. \tag{4.41}$$

This well-known fact [40] can be seen as follows: performing a pure spin-2 transformation using (4.41) we obtain

$$\begin{aligned}\delta^{(2)} e_n^{\mathcal{A}} &= D_n \left(e_r^{\mathcal{A}} \xi^r \right) \\ &= \xi^r D_r e_n^{\mathcal{A}} + e_r^{\mathcal{A}} \nabla_n \xi^r \\ &= \xi^r \nabla_r e_n^{\mathcal{A}} + e_r^{\mathcal{A}} \nabla_n \xi^r + f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} \left(\xi^r \omega_r^{\mathcal{B}} \right) e_n^{\mathcal{C}} ,\end{aligned} \tag{4.42}$$

where to obtain the second equality we have used the torsion constraint (4.19) and the third equality follows by the definition of the covariant derivative (4.18). The first two terms in the last line give the Lie derivative along ξ^n whereas the last term is a local Lorentz transformation (4.16a) with parameter $\Lambda^{\mathcal{B}} = \xi^r \omega_r^{\mathcal{B}}$. Since the metric-like fields are given by local Lorentz invariant combinations of generalized vielbeins (after solving the torsion constraint), we see that the choice (4.41) indeed generates the spin-2 gauge transformations of the metric-like fields. A generic $\Xi^{\mathcal{A}}$ introduces both a spin-2 and a spin-3 transformation, therefore there will be projections P and $1-P$ such that $P\Xi$ induces a pure spin-2 transformation. Instead of fixing the tensor $S^{\mathcal{A}}{}_{nn}$ of (4.40), we will rather fix the projector P. It should project an arbitrary gauge transformations to a pure spin-2 transformation. Therefore, we demand that

$$\text{for every } \Xi^{\mathcal{A}} \text{ there exists a } \xi^n \text{ such that } P^{\mathcal{A}}{}_{\mathcal{B}} \Xi^{\mathcal{B}} = S_n^{\mathcal{A}} \xi^n ,$$

and $P^2 = P$. It is natural to require that the projector is orthogonal with respect to the Killing form

$$P^{\mathcal{A}\mathcal{B}} = P^{\mathcal{B}\mathcal{A}} , \tag{4.43}$$

where we have raised the indices with the Killing form. This fixes the projector uniquely to be

$$P^{\mathcal{A}\mathcal{B}} = e_n^{\mathcal{A}} g^{nm} e_m^{\mathcal{B}} . \tag{4.44}$$

It indeed squares to itself

$$\begin{aligned}P^{\mathcal{A}}{}_{\mathcal{B}} \, P^{\mathcal{B}}{}_{\mathcal{C}} &= e_m^{\mathcal{A}} \, g^{mn} \, e_n^{\mathcal{D}} \, \kappa_{\mathcal{B}\mathcal{D}} \, e_r^{\mathcal{B}} \, g^{rp} \, e_p^{\mathcal{E}} \, \kappa_{\mathcal{E}\mathcal{C}} \\ &= e_m^{\mathcal{A}} \, g^{mn} \, g_{nr} \, g^{rp} \, e_p^{\mathcal{E}} \, \kappa_{\mathcal{E}\mathcal{C}} \\ &= e_m^{\mathcal{A}} \, g^{mp} \, e_p^{\mathcal{E}} \, \kappa_{\mathcal{E}\mathcal{C}} \\ &= P^{\mathcal{A}}{}_{\mathcal{C}} ,\end{aligned} \tag{4.45}$$

and projects an arbitrary gauge parameter $\Xi^{\mathcal{A}}$ to its spin-2 component as follows

$$P^{\mathcal{A}}{}_{\mathcal{B}} \, \Xi^{\mathcal{B}} = e_m^{\mathcal{A}} \left(g^{mn} \, e_n^{\mathcal{C}} \, \kappa_{\mathcal{C}\mathcal{B}} \, \Xi^{\mathcal{B}} \right) , \tag{4.46}$$

where we interpret the term in brackets as the corresponding vector field.

Having fixed P, we can now look for an ${S^{\mathcal{A}}}_{nr}$ that satisfies

$$ {P^{\mathcal{A}}}_{\mathcal{B}}\, {S^{\mathcal{B}}}_{nr} = 0 \ . \tag{4.47} $$

In addition, ${S^{\mathcal{A}}}_{nr}$ should reproduce the correct transformation behavior at the linear order

$$ {S^{\mathcal{A}}}_{nr} = 3\, {d^{\mathcal{A}}}_{bc}\, e^b_n\, e^c_r + \dots \tag{4.48} $$

A derivation of the fact that this choice leads to the correct free transformations can be found in Appendix A of our publication [25]. This fixes ${S^{\mathcal{A}}}_{nr}$ to be

$$ {S^{\mathcal{A}}}_{nr} = \left(\delta^{\mathcal{A}}_{\mathcal{D}} - P^{\mathcal{A}}_{\mathcal{D}}\right) 3\, {d^{\mathcal{D}}}_{\mathcal{BC}}\, e^{\mathcal{B}}_m\, e^{\mathcal{C}}_p \left(\delta^m_n\, \delta^p_r - \frac{1}{3} g^{mp}\, g_{nr}\right) , \tag{4.49} $$

where the projector in the last bracket ensures that only the traceless component of the gauge parameter ξ^{nn} contributes to a spin-3 gauge transformation also at the non-linear level.

4.4.2 *Spin-3 Transformations*

We are now in a position to determine the spin-3 transformations of the metric-like fields. To this end, we make the most general ansatz thereof to a given order and then fix its coefficients by applying the algorithm outlined earlier. This leads to the following result for the spin-3 transformation of the spin-3 field to cubic order

$$ \delta^{(3)}_{\Xi} \phi_{nnn} = \nabla_n \hat{\xi}_{nn} + (\hat{\xi}\phi\nabla\phi)_{nnn} + (\nabla\hat{\xi}\phi\phi)_{nnn} + \mathcal{O}(\phi^4) \ . \tag{4.50} $$

Here $\hat{\xi}_{nn}$ denotes the traceless component of ξ_{nn}. The second summand $(\hat{\xi}\phi\nabla\phi)$ above denotes terms with derivatives acting on the spin-3 fields whereas $(\nabla\hat{\xi}\phi\phi)$ stands for terms with derivatives on the gauge parameter. Their explicit form is quite involved and can be found in Appendix D of our publication [25]. Analogously, the spin-3 transformations of the metric to cubic order can be derived

$$ \begin{aligned} \delta^{(3)}_{\Xi} g_{nn} = 6\,(& 2\,\hat{\xi}^{rp}\,\nabla_r\phi_{nnr} + 4\,\hat{\xi}^{rp}\,g_{nn}\,\nabla_r\phi_r + \hat{\xi}_{nn}\,\nabla_r\phi^r \\ & - 2\,\hat{\xi}^{rp}\,g_{nn}\,\nabla_o{\phi_{rp}}^o - 2\,{\hat{\xi}_n}{}^r\,\nabla_n\phi_r - 2\,{\hat{\xi}_n}{}^r\,\nabla_r\phi_n \\ & + 2\,{\hat{\xi}_n}{}^r\,\nabla^m\phi_{nrm} - 2\,\hat{\xi}^{rp}\,\nabla_n\phi_{n)rp}) + (\hat{\xi}\phi\phi\nabla\phi)_{nn} + \mathcal{O}(\phi^5)\,, \end{aligned} $$

where $(\hat{\xi}\phi\phi\nabla\phi)$ can again be found in Appendix D of our publication [25].

The determined spin-3 transformations of the metric-like fields are unfortunately quite lengthy. Having obtained the metric-like gauge variations to this order, we could now also fix the corresponding action. Using a computer algebra program this would be a straightforward task. However, it is to be expected that the action would be of similar size and we therefore did not attempt to do so. On the other hand, one can try to determine the algebra of the gauge transformations which we just found. As we will see in the following sections, this will lead to expressions of more manageable size.

4.5 GAUGE ALGEBRA

In this section, we will study the gauge algebra of the metric-like fields. In general, gauge algebras are interesting as they allow one to deduce the structure of the underlying symmetry algebra. Furthermore, it was observed in [46] that the gauge algebra of the metric-like fields only closes on-shell while the gauge algebra of the frame-like fields closes off-shell. In the following section, we will explain how this seeming discrepancy arises. This will also lead us to a procedure to calculate the gauge algebra, which is more efficient than to evaluate the commutator of the metric-like gauge transformations discussed in the last section.

4.5.1 *On-shell Gauge Algebra*

Recall from Section 4.4.1 that in the frame-like theory general local translations, which we take to be parameterized by $\Xi^{\mathcal{A}}$, induce both spin-2 and spin-3 transformations. According to (4.17), the frame-like fields transform as follows

$$\delta_\Xi e_m^{\mathcal{A}} = D_m \Xi^{\mathcal{A}} \,, \tag{4.51}$$

$$\delta_\Xi \omega_m^{\mathcal{A}} = \frac{1}{l^2} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, e_m^{\mathcal{B}}\, \Xi^{\mathcal{C}} \,, \tag{4.52}$$

and the gauge algebra closes off-shell.

When we translate the frame-like to the metric-like theory, we have to solve the torsion constraint (4.19) to express the spin-connection in terms of vielbeins, $\omega = \omega(e)$. This implicit dependence induces a gauge transformation of the spin-connection that differs from the transformation (4.52) and only coincides with it on-shell, i.e. after using the equations of motion. This can be seen as follows: the induced transformation of the spin-connection can be calculated by varying the torsion constraint (4.19),

$$\delta_\Xi \left(D_{[m} e_{n]}^{\mathcal{A}} \right) = \delta_\Xi D_{[m} e_{n]}^{\mathcal{A}} + D_{[m} D_{n]} \Xi^{\mathcal{A}} = 0 \,. \tag{4.53}$$

The Christoffel symbol is symmetric in n and m and therefore the variation of the covariant derivative in the equation above is given by the transformation of the spin-connection. We thus obtain

$$f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, \delta_\Xi \omega_{[m}^{\mathcal{B}}\, e_{n]}^{\mathcal{C}} + f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, R_{mn}^{\mathcal{B}}\, \Xi^{\mathcal{C}} = 0 \tag{4.54}$$

where $R_{mn}^{\mathcal{A}}$ is the generalized curvature tensor (4.15). The equation of motion for the vielbein is

$$R_{mn}^{\mathcal{A}} = -\frac{1}{2l^2} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, e_m^{\mathcal{B}}\, e_n^{\mathcal{C}} \,. \tag{4.55}$$

Using this equation of motion in (4.54), we find that the induced transformation reduces on-shell to (we assume that the vielbein is non-degenerate)

$$\delta_\Xi \omega_m^{\mathcal{A}} = \frac{1}{l^2} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, e_m^{\mathcal{B}}\, \Xi^{\mathcal{C}} \,, \tag{4.56}$$

which coincides with the transformation (4.52) in the frame-like theory. Therefore, we expect that the metric-like gauge algebra only closes on-shell.

Let us explicitly consider the commutator of two gauge transformations of a vielbein (all metric-like fields are built out of vielbeins). Using (4.51) we obtain

$$\begin{aligned}[][\delta_\Xi, \delta_\Pi] e_m^{\mathcal{A}} &= D_m \left(\delta_\Xi \Pi^{\mathcal{A}} - \delta_\Pi \Xi^{\mathcal{A}}\right) + (\delta_\Xi D_m)\Pi^{\mathcal{A}} - (\delta_\Pi D_m)\Xi^{\mathcal{A}} \\ &= \delta_{(\delta_\Xi \Pi - \delta_\Pi \Xi)} e_m^{\mathcal{A}} + f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} \left(\delta_\Xi \omega_m^{\mathcal{B}}\, \Pi^{\mathcal{C}} - \delta_\Pi \omega_m^{\mathcal{B}}\, \Xi^{\mathcal{C}}\right) \ .\end{aligned} \tag{4.57}$$

The first term is a local translation of the generalized vielbein and therefore can again be interpreted as a gauge transformation in the metric-like formulation. The second term can in general not be rewritten as a gauge transformation of the vielbein.[4] On the other hand, on-shell the last term is a generalized local Lorentz transformation of the generalized vielbein as can be checked by using (4.56) and the Jacobi identity

$$\begin{aligned} & f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} \left(\delta_\Xi \omega_m^{\mathcal{B}}\, \Pi^{\mathcal{C}} - \delta_\Pi \omega_m^{\mathcal{B}}\, \Xi^{\mathcal{C}}\right) \\ &\quad = -\frac{1}{l^2} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} \left(f^{\mathcal{B}}{}_{\mathcal{D}\mathcal{E}}\, e_m^{\mathcal{D}}\, \Xi^{\mathcal{E}}\, \Pi^{\mathcal{C}} - f^{\mathcal{B}}{}_{\mathcal{D}\mathcal{E}}\, e_m^{\mathcal{D}}\, \Pi^{\mathcal{E}}\, \Xi^{\mathcal{C}}\right) \\ &\quad = \frac{1}{l^2} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{D}} \left(f^{\mathcal{B}}{}_{\mathcal{E}\mathcal{C}}\, \Xi^{\mathcal{E}}\, \Pi^{\mathcal{C}}\right) e_m^{\mathcal{D}} \ . \end{aligned} \tag{4.58}$$

For metric-like fields, all frame indices are contracted with invariant tensors, and therefore the local Lorentz transformations do not have any effect. Hence we find that on-shell the gauge algebra in the metric-like formulation is obtained by translating

$$[\delta_\Xi, \delta_\Pi] = \delta_{(\delta_\Xi \Pi - \delta_\Pi \Xi)} \tag{4.59}$$

to metric-like quantities. This provides us with a very efficient way to determine the gauge algebra as we will discuss in the following.

4.5.2 *Spin-2 Spin-2 Commutator*

We now consider the case of both transformations being diffeomorphisms, i.e. $\Pi^{\mathcal{A}} = e_m^{\mathcal{A}} \pi^m$ and $\Xi^{\mathcal{A}} = e_m^{\mathcal{A}} \xi^m$. Using (4.57), we can calculate the resulting transformation

$$\begin{aligned} \delta_\Pi \left(e_m^{\mathcal{A}}\, \xi^m\right) - \delta_\Xi \left(e_m^{\mathcal{A}}\, \pi^m\right) &= D_m \left(e_n^{\mathcal{A}}\, \pi^n\right) \xi^m - \xi \leftrightarrow \pi \\ &= -e_n^{\mathcal{A}}\, \mathcal{L}_\pi \xi^n + \xi^m\, \pi^n\, D_{[m} e_{n]}^{\mathcal{A}} \end{aligned} \tag{4.60}$$

where $\mathcal{L}_\pi \xi^n = \pi^m \partial_m \xi^n - \xi^m \partial_m \pi^n$ is the Lie derivative. The last term in (4.60) vanishes after imposing the torsion constraint (4.19). By (4.41), the result of this commutator therefore induces a diffeomorphism with vector field $-\mathcal{L}_\pi \xi^n$.

4 It can be shown that this term reduces to a local Lorentz rotation for the case of at least one spin-2 gauge transformation. We will not reproduce the proof here. It can be found in Section 4.2 of our publication [25].

4.5.3 *Spin-3 Spin-2 Commutator*

We will now discuss the commutator of a spin-3 and a spin-2 transformation. The spin-3 transformation is parameterized by

$$\Xi^{\mathcal{A}} = S^{\mathcal{A}}{}_{nm}\,\xi^{nm}\ , \tag{4.61}$$

where S is given in (4.49). The result for the commutator will not depend on the precise form of S, but only on the property that it is built from the vielbeins. In fact, we can also consider the more general case of the commutator of a spin-$(s+1)$ and a spin-2 transformation without any additional complication. The spin-2 and the spin-$(s+1)$ transformations are parameterized by

$$\Pi^{\mathcal{A}} = e^{\mathcal{A}}_m \pi^m \quad \text{and} \quad \Xi^{\mathcal{A}} = S^{\mathcal{A}}{}_{m(s)}\,\xi^{m(s)}\ . \tag{4.62}$$

Here, $S^{\mathcal{A}}{}_{m(s)}$ is built by contracting vielbeins and it is completely symmetric in all space-time indices. For later purposes we consider the following space-time tensor,

$$\mathcal{O}_{nm(s)} = \kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n\, S^{\mathcal{B}}{}_{m(s)}\ . \tag{4.63}$$

Because this tensor is constructed from the vielbeins, it will transform under the spin-2 transformation by the Lie derivative along π

$$\delta_\Pi \mathcal{O}_{nm(s)} = \pi^p \nabla_p \mathcal{O}_{nm(s)} + (\nabla_m \pi^p)\,\mathcal{O}_{npm(s-1)} + (\nabla_n \pi^p)\,\mathcal{O}_{pm(s)}\ . \tag{4.64}$$

The left hand side of this equation can be calculated by explicitly evaluating the variation of the vielbein, i.e.

$$\begin{aligned}
\delta_\Pi \mathcal{O}_{nm(s)} &= \kappa_{\mathcal{AB}}\, D_n(e^{\mathcal{A}}_p \pi^p) S^{\mathcal{B}}{}_{m(s)} + \kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n (\delta_\Pi S^{\mathcal{B}}{}_{m(s)}) \\
&= \kappa_{\mathcal{AB}}\, \pi^p (D_p e^{\mathcal{A}}_n) S^{\mathcal{B}}{}_{m(s)} + \kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n (\delta_\Pi S^{\mathcal{B}}{}_{m(s)}) \\
&\quad + (\nabla_n \pi^p)\,\mathcal{O}_{pm(s)}\ ,
\end{aligned} \tag{4.65}$$

where we used the torsion constraint (4.19). Combining (4.64) with (4.65) leads to

$$\kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n (\delta_\Pi S^{\mathcal{B}}{}_{m(s)}) = \kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n\, \pi^p D_p S^{\mathcal{B}}{}_{m(s)} + \kappa_{\mathcal{AB}}\, e^{\mathcal{A}}_n\, (\nabla_m \pi^p)\, S^{\mathcal{B}}{}_{pm(s-1)}\ .$$

We therefore conclude that

$$\delta_\Pi S^{\mathcal{B}}{}_{m(s)} = \pi^p D_p S^{\mathcal{B}}{}_{m(s)} + (\nabla_m \pi^p)\, S^{\mathcal{B}}{}_{pm(s-1)}\ . \tag{4.66}$$

We are now in the position to determine the commutator of the spin-2 transformation Π and the spin-$(s+1)$ transformation Ξ given in (4.62), and we find

$$\begin{aligned}
\delta_\Xi \Pi^{\mathcal{A}} - \delta_\Pi \Xi^{\mathcal{A}} &= \pi^p D_p \left(S^{\mathcal{A}}{}_{m(s)} \xi^{m(s)}\right) - \xi^{m(s)}\ \delta_\Pi S^{\mathcal{A}}{}_{m(s)} \\
&= S^{\mathcal{A}}{}_{m(s)} \left(\pi^p \nabla_p \xi^{\mu(s)} - \xi^{pm(s-1)} \nabla_p \pi^m\right) \\
&= S^{\mathcal{A}}{}_{m(s)} \left(\mathcal{L}_\pi \xi^{m(s)}\right)\ .
\end{aligned} \tag{4.67}$$

Thus, the commutator is a spin-$(s+1)$ transformation whose parameter is given by the Lie derivative of the original spin-$(s+1)$ parameter. In particular, for the case of spin-3, we find

$$[\delta^{(3)}_{\xi}, \delta^{(2)}_{\pi}] = \delta^{(3)}_{\mathcal{L}_{\pi}\xi} \ . \tag{4.68}$$

4.5.4 *Spin-3 Spin-3 Commutator*

In contrast to the commutation relation involving at least one spin-2 transformation, we were not able to derive an all-order result for the commutator of two spin-3 transformations. The commutator is specified by traceless parameters $\hat{\xi}^{nm}$ and $\hat{\pi}^{nm}$, and generically it will lead to a combination of a spin-2 transformation and a spin-3 transformation, i.e.

$$[\delta_{\Pi}, \delta_{\Xi}]\, e^{\mathcal{A}}_{m} = \delta_{S(u,v)}\, e^{\mathcal{A}}_{m} \ , \tag{4.69}$$

where

$$S^{\mathcal{A}}(u,v) = S^{\mathcal{A}}{}_{m} v^{m} + S^{\mathcal{A}}{}_{nm} u^{nm} \ . \tag{4.70}$$

In the following, we will determine the parameters u^{nm} and v^{m} perturbatively in the spin-3 field. The spin-2 parameter v^{m} was already calculated in [46] considering zeroth order contributions and we reproduced this result in [25] using a different method. It is given by

$$v^{m} = -18\, g^{mn} \left(\xi^{rp}\, \nabla_{n} \pi_{rp} - \frac{1}{3}\, \xi^{r}{}_{r}\, \nabla_{n} \pi^{p}{}_{p} - \xi \leftrightarrow \pi \right) \ . \tag{4.71}$$

We will use the algorithm discussed in Section 4.3 to determine the parameter u^{nm} at linear order. To this end, we consider

$$\delta^{(3)}_{\Pi}\, \Xi^{\mathcal{A}} - \delta^{(3)}_{\Xi}\, \Pi^{\mathcal{A}} = S^{\mathcal{A}}(u^{mn}, v^{m}) \ , \tag{4.72}$$

where $S^{\mathcal{A}}$ was defined in (4.70). The result (4.71) for the parameter v^{n} cannot be corrected by terms linear in the spin-3 field as we cannot build a vector by contracting a spin-3 field, a covariant derivative and the parameter u^{mn}. For the linear order of u^{mn}, we make the most general ansatz containing all possible contractions of

$$\xi^{rp}\ , \ \pi^{rp} \text{ and } \phi_{pru} \tag{4.73}$$

with two symmetric free indices, m and n, and antisymmetric with respect to the exchange of ξ and π. We plug this ansatz in (4.72) and use the algorithm described in Section 4.3 to determine its coefficients. The result for u^{nm} contains three different contributions denoted by

$$u^{nm} = u^{nm}_{1} + u^{nm}_{2} + u^{nm}_{3} \ . \tag{4.74}$$

The first summand u_1^{nm} contains all terms with a derivative acting on the spin-3 field

$$\begin{aligned}u_1^{nn} = 3\Big(&- 10\,(\hat{\xi}\hat{\pi})^{nnpq}\nabla_q\phi_p + 2\,(\hat{\xi}\hat{\pi})^{pq}{}_p{}^o(g^{nn}\nabla_o\phi_q - \nabla_o\phi^{nn}{}_q)\\ &+ 6\,(\hat{\xi}\hat{\pi})^{nnpq}\nabla_o\phi_{pq}{}^o - 12\,(\hat{\xi}\hat{\pi})^{pqor}g^{nn}\nabla_r\phi_{pqo}\\ &+ 3\,(\hat{\xi}\hat{\pi})^{np}{}_p{}^q\nabla^n\phi_q + 2\,(\hat{\xi}\hat{\pi})^{np}{}_p{}^q\nabla_q\phi^n\\ &- 3\,(\hat{\xi}\hat{\pi})^{np}{}_p{}^q\nabla^o\phi^n{}_{qo} - 5\,(\hat{\xi}\hat{\pi})^{npqo}\nabla_p\phi^n{}_{qo}\\ &+ 13\,(\hat{\xi}\hat{\pi})^{npqo}\nabla_q\phi^n{}_{po}\Big)\ ,\end{aligned}$$

where we have used the following notation

$$(\hat{\xi}\hat{\pi})^{mnrp} = \hat{\xi}^{mn}\hat{\pi}^{rp} - \hat{\xi} \leftrightarrow \hat{\pi}\ . \tag{4.75}$$

The hatted tensors again denote the traceless components of the parameters. The second summand collects all contributions with a derivative acting on one of the parameters,

$$\begin{aligned}u_2^{nn} = 12\,\Big(&(\nabla\hat{\xi}\hat{\pi})^{qnn}{}_{pq}\phi^p + (\nabla\hat{\xi}\hat{\pi})^q{}_p{}^o{}_{qo}\phi^p g^{nn} - (\nabla\hat{\xi}\hat{\pi})^{pqo}{}_{pq}\phi^{nn}{}_o\\ &- \tfrac{3}{2}\,(\nabla\hat{\xi}\hat{\pi})^{pnnqo}\phi_{pqo} + 12\,(\nabla\hat{\xi}\hat{\pi})^{pqo}{}_p{}^r g^{nn}\phi_{qor}\\ &- \tfrac{1}{2}\,(\nabla\hat{\xi}\hat{\pi})^{pnqn}{}_p\phi_q - \tfrac{1}{2}\,(\nabla\hat{\xi}\hat{\pi})^{pnq}{}_{pq}\phi^n\\ &-(\nabla\hat{\xi}\hat{\pi})^{pnq}{}_p{}^o\phi^n{}_{qo} - 17\,(\nabla\hat{\xi}\hat{\pi})^{pqon}{}_p\phi^n{}_{qo}\Big)\ .\end{aligned}$$

Here we defined

$$(\nabla\hat{\xi}\hat{\pi})_m{}^{nrpo} = \hat{\pi}^{nr}\nabla_m\hat{\xi}^{po} - \hat{\pi} \leftrightarrow \hat{\xi}\ . \tag{4.76}$$

Finally, there are contributions containing the trace of the parameters of the gauge transformations

$$\begin{aligned}u_3^{nn} = 4\,&(\hat{\xi}\pi' - \hat{\pi}\xi')^{ro}\nabla_o\phi^{nn}{}_r + 4\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{ro}g^{nn}\nabla_o\phi_r\\ &+ 2\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{nn}\nabla_o\phi^o - 4\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{nr}\nabla^n\phi_r\\ &- 4\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{(nr}\nabla_r\phi^n + 4\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{nr}\nabla^o\phi^n{}_{ro}\\ &- 4\,(\hat{\xi}\pi' - \hat{\pi}\xi')^{ro}\nabla^n\phi^n{}_{ro}\ ,\end{aligned}$$

where we denoted

$$(\hat{\xi}\pi' - \hat{\pi}\xi')^{nm} = \hat{\xi}^{mn}\pi^p{}_p - \hat{\pi}^{mn}\xi^p{}_p\ . \tag{4.77}$$

It might at first seem surprising that the commutator contains traces of the gauge parameters, whereas in a single gauge transformation only their traceless part contributes. However, this is due to the fact that the notion of the trace is field-dependent (it depends on the metric), and that the field changes under the gauge transformations.

We have therefore calculated the commutator of two spin-3 transformations at linear order in the spin-3 field. Together with the expression derived for the commutator of a spin-2 with either a spin-2 or spin-3 transformation, which are exact results, we have determined the gauge algebra to leading order.

4.6 SUMMARY

This chapter started by discussing that Vasiliev equations for vanishing zero-form can be described by a straightforward generalization of the Chern–Simons action for three-dimensional gravity. We furthermore saw that for $\lambda = 3$ the theory only contains spin-2 and spin-3 gauge fields. Inspired by gravity, we then wanted to understand the underlying geometry of this particularly simple higher-spin theory which led us to rewrite it in terms of metric-like quantities. To this end, we developed an efficient algorithm to translate frame-like to metric-like expressions (perturbatively in the spin-3 fields). We chose to first determine the gauge transformations of the metric-like fields. In order to do so, a relation between the frame-like and the metric-like fields was required, for which we followed the results of [15]. We also required a map between frame-like and metric-like gauge parameters. We found a particularly natural map which is consistent to arbitrary order in the spin-3 fields. Unfortunately, the resulting gauge transformations are quite involved and we could not find a pattern that would allow us to organize them in a more manageable form. We therefore chose not to determine the corresponding action from the gauge transformations. While this would be a straightforward task using a computer algebra program, the result would be of similar size and therefore of limited practical use. Instead, we considered the gauge algebra and we found exact expressions for the commutators involving spin-2 gauge transformations. However, we were not able to derive a closed expression for the commutator of two spin-3 transformations and we therefore determined the resulting gauge algebra to leading order in the spin-3 fields.

Concluding, it is fair to say that a geometrical understanding of three-dimensional Vasiliev theory, even in this most simple incarnation, remains elusive. While we were able to clarify a number of questions left unanswered by [46] and developed an algorithm which efficiently translates frame-like to metric-like quantities, our findings also show that one requires an organizing principle or pattern underlying the metric-like expressions in order to make practical use of them.

5 SECOND ORDER ANALYSIS

In this chapter, we will extract the second order equations of motion for twisted and physical fields from Vasiliev equations. First we will discuss the delicate issue of local Lorentz covariance within Vasiliev theory. We will then study whether a field frame can be found in which all second order twisted fields can be set to zero and extract the second order equations of motion for physical fields in this field frame. This chapter will close by briefly discussing the important question of physically allowed field-redefinitions within Vasiliev theory.

5.1 GENERALITIES

Let us briefly recall how equations of motion for the physical and twisted fields can be extracted from Vasiliev theory.

As a first step, one solves the non-dynamical Vasiliev equations (3.105c)-(3.105e) to determine the z-dependence of the masterfields by using the homotopy integrals (3.110). Formally, this then leads to

$$\mathcal{W} = \boldsymbol{\omega}(y,\phi,\psi) + z^\alpha \Gamma_0 \langle D_\Omega \mathcal{A}_\alpha - [\mathcal{W},\mathcal{A}_\alpha]_\star \rangle \,, \tag{5.1a}$$

$$\mathcal{B} = \boldsymbol{C}(y,\phi,\psi) + z^\alpha \Gamma_0 \langle [\mathcal{A}_\alpha,\mathcal{B}]_\star \rangle \,, \tag{5.1b}$$

$$\mathcal{A}_\alpha = z_\alpha \Gamma_1 \langle \mathcal{A}_\nu \star \mathcal{A}^\nu + \mathcal{B} \star \varkappa \rangle \,, \tag{5.1c}$$

where we have used that upon imposing the Schwinger–Fock gauge (3.113) the homogeneous part of the form $\partial^z_\alpha \boldsymbol{\epsilon}(y,z,\phi,\psi)$ in $\mathcal{A}_\alpha$ will not be present.

One then inserts these expressions in the dynamical Vasiliev equations (3.105a)-(3.105b). At order n in perturbation theory one obtains

$$D_\Omega \boldsymbol{C}^{(n)} = -D_\Omega \left(z^\alpha \Gamma_0 \langle [\mathcal{A}_\alpha,\mathcal{B}]_\star \rangle \right) \Big|^{(n)} + [\mathcal{W},\mathcal{B}]_\star \Big|^{(n)} \,, \tag{5.2a}$$

$$D_\Omega \boldsymbol{\omega}^{(n)} = -D_\Omega \left(z^\alpha \Gamma_0 \langle D_\Omega \mathcal{A}_\alpha - [\mathcal{W},\mathcal{A}_\alpha]_\star \rangle \right) \Big|^{(n)} + \mathcal{W} \wedge \star \mathcal{W} \Big|^{(n)} \,, \tag{5.2b}$$

where the right hand side of the equations involves z-dependent terms and $|^{(n)}$ denotes the contribution of the n-th order in perturbation theory. Since the left hand sides of the equations above are z-independent, their right hand sides have to have the same property. Therefore, we can evaluate the right hand sides of the equations above by first performing all the star products and then setting $z = 0$ as all z-dependence has to cancel out.

The resulting expressions are the equations of motion for the fields $\boldsymbol{C}^{(n)}$ and $\boldsymbol{\omega}^{(n)}$. By decomposing them further in physical and twisted fields

$$\boldsymbol{C}^{(n)} = \hat{C}^{(n)}\psi + \tilde{C}^{(n)}\,, \tag{5.3}$$

$$\boldsymbol{\omega}^{(n)} = \hat{\omega}^{(n)} + \tilde{\omega}^{(n)}\psi\,, \tag{5.4}$$

one then obtains their equations of motion at n-th in perturbation theory.

Following this recipe, we obtained the linear equations of motion in Chapter 3. In this chapter, we will now extract the second order equations.

5.2 MANIFEST LORENTZ COVARIANCE

As we will discuss now, the Vasiliev equations (3.105) will not lead to manifestly local Lorentz covariant equations of motion for the physical and twisted fields.

At second order, the one-form equations of motion obtained from Vasiliev theory are of the form

$$\nabla\boldsymbol{\omega}^{(2)} = \bar{\omega}^{\alpha}{}_{\beta} \wedge \bar{\omega}^{\alpha\beta}\, T_{\alpha\alpha}(\hat{C}^{(1)}, \hat{C}^{(1)}) + \dots\,, \tag{5.5}$$

where we only made contributions explicit involving the background spin-connection. We derive this fact and the explicit form of $T_{\alpha\alpha}$ in Appendix B.4. The term on the right hand side obviously contains "naked" background spin-connections, i.e. spin-connections which are not contained within Lorentz covariant derivatives. Therefore, the equations of motion derived from Vasiliev equations at second order are indeed not manifestly local Lorentz covariant.

As was first shown[1] in [2], one can ensure that no such "naked" background spin-connections appear by performing following field redefinition

$$\bar{\omega}^{\alpha\alpha}\, L^{y}_{\alpha\alpha} \;\rightarrow\; \bar{\omega}^{\alpha\alpha}(L^{y}_{\alpha\alpha} + L^{z}_{\alpha\alpha} - L^{s}_{\alpha\alpha}) = \bar{\omega}^{\alpha\alpha}\hat{L}_{\alpha\alpha}\,. \tag{5.6}$$

Here we have used the following definitions

$$L^{y}_{\alpha\beta} = -\frac{i}{4}\{y_\alpha, y_\beta\}_\star\,, \quad L^{z}_{\alpha\beta} = \frac{i}{4}\{z_\alpha, z_\beta\}_\star\,, \quad L^{s}_{\alpha\beta} = \frac{i}{4}\{\mathcal{S}_\alpha, \mathcal{S}_\beta\}_\star\,,$$

from which one can construct

$$L^{0}_{\alpha\beta} = L^{y}_{\alpha\beta} + L^{z}_{\alpha\beta}\,, \tag{5.7}$$

$$\hat{L}_{\alpha\beta} = L^{0}_{\alpha\beta} - L^{s}_{\alpha\beta}\,. \tag{5.8}$$

The vacuum masterfields are given in terms of the redefined spin connection by

$$\Omega = \frac{1}{2}\bar{\omega}^{\alpha\beta}\hat{L}_{\alpha\beta} + \frac{1}{2}\bar{e}^{\alpha\beta}P_{\alpha\beta}\,, \qquad \mathcal{S}^{(0)}_{\alpha} = z_\alpha\,, \qquad \mathcal{B}^{(0)} = 0\,. \tag{5.9}$$

1 We also refer to [3, 28, 53, 54] for related discussions of the four-dimensional case.

To see the effect of this field redefinition, we consider modified Vasiliev equations obtained by shifting every field by its respective vacuum value (5.9) (similar to (3.105) but now in terms of the redefined background spin connection). Following the same steps used before to derive (3.105) from (3.94), we obtain

$$D^{yz}\mathcal{W} = \mathcal{W}\wedge\star\mathcal{W} - \tfrac{1}{2}E^{\alpha\alpha}L^{s}_{\alpha\alpha} + \chi\,, \tag{5.10a}$$

$$D^{yz}\mathcal{B} = [\mathcal{W},\mathcal{B}]_\star\,, \tag{5.10b}$$

$$\partial^z_\alpha\mathcal{W} = [\mathcal{A}_\alpha, \bar{e}+\mathcal{W}]_\star + \chi_\alpha\,, \tag{5.10c}$$

$$\partial^z_\alpha\mathcal{B} = [\mathcal{A}_\alpha,\mathcal{B}]_\star\,, \tag{5.10d}$$

$$\partial^z_\alpha\mathcal{A}^\alpha = \mathcal{A}_\alpha\star\mathcal{A}^\alpha + \mathcal{B}\star\varkappa\,, \tag{5.10e}$$

where χ_α obeys $z^\alpha\chi_\alpha = 0$. As a result, χ_α will not contribute when we solve (5.10c) using the homotopy integral (3.110b) and we have therefore not given its explicit form. Similarly, χ vanishes for $z = 0$ and thus its explicit form is also of no importance because we will evaluate the dynamical equations at $z = 0$. Furthermore, the covariant derivative D^{yz} is given by

$$D^{yz}\bullet := \mathrm{d}\bullet - \frac{1}{2}\bar{\omega}^{\alpha\alpha}[L^0_{\alpha\alpha},\bullet]_\star - \frac{1}{2}\bar{e}^{\alpha\alpha}[P_{\alpha\alpha},\bullet]_\star\,. \tag{5.11}$$

To show that the equations (5.10) indeed do not lead to any "naked" background spin-connections, we formally solve for the z-dependence of the masterfields using the last three equations in (5.10). By using the homotopy integrals (3.110), one formally obtains

$$\mathcal{W} = \boldsymbol{\omega}(y,\phi,\psi) - z^\nu\Gamma_0\langle[\bar{e},\mathcal{A}_\nu]_\star + [\mathcal{W},\mathcal{A}_\nu]_\star\rangle\,, \tag{5.12a}$$

$$\mathcal{B} = \boldsymbol{C}(y,\phi,\psi) + z^\alpha\Gamma_0\langle[\mathcal{A}_\alpha,\mathcal{B}]_\star\rangle\,, \tag{5.12b}$$

$$\mathcal{A}_\alpha = z_\alpha\Gamma_1\langle\mathcal{A}_\nu\star\mathcal{A}^\nu + \mathcal{B}\star\varkappa\rangle\,. \tag{5.12c}$$

We observe that (5.12) (and in particular (5.12a) in stark contrast to (5.1a)) does not contain the background spin-connection $\bar{\omega}$ on its right hand sides. Therefore, no background spin-connection will enter the dynamical equations through the z-dependence of the masterfields.

The next step is to insert (5.12) in the first two redefined Vasiliev equations (5.10) and evaluate them at vanishing z

$$D^{yz}\mathcal{W}\Big|_{z=0} = \mathcal{W}\wedge\star\mathcal{W}\Big|_{z=0} - \tfrac{1}{2}E^{\alpha\alpha}L^s_{\alpha\alpha}\Big|_{z=0} \tag{5.13a}$$

$$D^{yz}\mathcal{B}\Big|_{z=0} = [\mathcal{W},\mathcal{B}]_\star\Big|_{z=0}\,. \tag{5.13b}$$

The only dependence in these equations on the background spin connection is through the covariant derivative D^{yz}. Up to terms proportional to the background vielbein, this covariant derivative leads to

$$\left(\mathrm{d}\mathcal{W} - \frac{1}{2}\bar{\omega}^{\alpha\alpha}[L^0_{\alpha\alpha},\mathcal{W}]_\star\right)_{z=0} = \nabla\boldsymbol{\omega}(y)\,, \tag{5.14}$$

and analogously for $\mathcal{B}$. Here the Lorentz covariant derivative is defined as before, i.e. $\nabla = \mathrm{d} - \bar{\omega}^{\alpha\alpha} y_\alpha \partial^y_\alpha$. This can be seen by observing[2]

$$[L^0_{\alpha\alpha}, \bullet]_\star \Big|_{z=0} = (y_\alpha \partial^y_\alpha + z_\alpha \partial^z_\alpha) \bullet \Big|_{z=0} = y_\alpha \partial^y_\alpha \bullet \Big|_{z=0} . \tag{5.15}$$

We have thus shown that for equations of motion obtained from the redefined Vasiliev equations (5.10) the background spin-connection only appears as part of the Lorentz covariant derivative ∇. This implies that the resulting physical and twisted equations of motion are manifestly local Lorentz covariant with respect to the background fields to arbitrary order in perturbation theory.

One can also ensure manifest local Lorentz covariance with respect to the full spin-connection $\omega^{\alpha\alpha}$ by performing an analogous field redefinition

$$\omega^{\alpha\beta} L^y_{\alpha\alpha} \to \omega^{\alpha\alpha} \hat{L}_{\alpha\alpha} . \tag{5.16}$$

But we will not do so in the following as we will be mostly interested in terms for which only background spin-connections appear.[3] We refer to our publication [26] for more details on this last point.

5.3 LINEAR ORDER

The linear equations for the physical and twisted fields extracted from the redefined Vasiliev equations (5.10) agree with the ones extracted from the original set of equations (3.105).

To see this, we determine the z-dependence of $\mathcal{W}^{(1)}$ using (5.10c)

$$\mathcal{W}^{(1)} = \boldsymbol{\omega}^{(1)}(y) - z^\alpha \Gamma_0 \langle [\bar{e}, \mathcal{A}^{(1)}_\alpha]_\star \rangle . \tag{5.17}$$

Therefore, the masterfield $\mathcal{W}^{(1)}$ after the redefinition can be obtained from the previous one (3.112c) by dropping all terms proportional to the background spin-connection.[4] Evaluation of the star products will lead to

$$\mathcal{W}^{(1)} = \hat{\omega}^{(1)}(y) + \tilde{\omega}^{(1)}(y)\psi + M_2\psi + \tilde{M}_2 , \tag{5.18}$$

where we have used the result (3.115a) but dropped all terms proportional to the background spin-connection. We then insert this result for

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2 This needs to be contrasted with

$$[L^y_{\alpha\alpha}, \bullet]_\star \Big|_{z=0} = (y_\alpha + \partial^z_\alpha) \partial^y_\alpha \bullet \Big|_{z=0} ,$$

which appeared through D_Ω in the Vasiliev equations (3.105) before the redefinition.

3 We will extract second order corrections to the free equations bilinear in the scalar fields. Therefore only the background spin-connection may appear in such correction terms.

4 The expression (3.112c) can be rewritten as

$$\mathcal{W}^{(1)} = \boldsymbol{\omega}^{(1)}(y) - z^\alpha \Gamma_0 \langle [\bar{\omega} + \bar{e}, \mathcal{A}^{(1)}_\alpha]_\star \rangle ,$$

because of $z^\alpha \mathrm{d}\mathcal{A}^{(1)}_\alpha = 0$ in Schwinger–Fock gauge.

$\mathcal{W}^{(1)}$ in the corresponding dynamical Vasiliev equation (5.10a). In this step, it is crucial that at linear order[5]

$$L^s_{\alpha\alpha}|_{z=0} = 0\,. \tag{5.19}$$

Using these results, one can then easily check that the linear equations of motion for the physical and twisted one-form are unchanged by the redefinition of the background spin-connection and are therefore given by (3.118).

For the second order analysis, we will choose vanishing solutions for the twisted zero and one-form

$$\tilde{C}^{(1)} = 0\,, \qquad\qquad \tilde{\omega}^{(1)} = 0\,. \tag{5.20}$$

This can only be done after having performed the field redefinition $\mathcal{W}^{(1)} \to \mathcal{W}^{(1)} + M_1\psi$ of (3.123) to cancel the source term to the twisted one-form. We therefore have

$$\mathcal{W}^{(1)} = \hat{\omega}^{(1)}(y) + M_2\psi + M_1\psi =: \hat{\omega}^{(1)}(y) + M\psi\,, \tag{5.21}$$

where we have used that $\tilde{M}_2$ vanishes for $\tilde{C}^{(1)} \equiv 0$. The explicit expressions for M_1 and M_2 are given in (3.123) and (3.116a) and imply that

$$\begin{aligned} M = \frac{\phi}{2}\bar{e}^{\alpha\alpha}\int_0^1 \mathrm{d}t\,(t-1) \\ \times\Big\{\, e^{ityz}\, z_\alpha \left(y_\alpha(1-t) - i(1+t)t^{-1}\partial^z_\alpha\right)\hat{C}^{(1)}(-zt) \\ + \tfrac{1}{2}(t+1)\left(g_0\, y_\alpha y_\alpha + 2iy_\alpha t^{-1}\partial^y_\alpha - g_0\, t^{-2}\partial^y_\alpha\partial^y_\alpha\right)\hat{C}^{(1)}(ty)\Big\}\,. \end{aligned}$$

This solution obviously contains the free parameter g_0 which will therefore enter the second order equations discussed in the following.

5.4 SECOND ORDER

We start by considering the masterfields (5.12) at the second order. In Schwinger–Fock gauge (3.113) they are given by

$$\mathcal{W}^{(2)} = \boldsymbol{\omega}^{(2)}(y,\phi,\psi) - z^\nu\Gamma_0\langle[\bar{e},\mathcal{A}^{(2)}_\nu]_\star + [\mathcal{W}^{(1)},\mathcal{A}^{(1)}_\nu]_\star\rangle\,, \tag{5.22a}$$

$$\mathcal{B}^{(2)} = \boldsymbol{C}^{(2)}(y,\phi,\psi) + z^\alpha\Gamma_0\langle[\mathcal{A}^{(1)}_\alpha,\mathcal{B}^{(1)}]_\star\rangle\,, \tag{5.22b}$$

$$\mathcal{A}^{(2)}_\alpha = z_\alpha\Gamma_1\langle\mathcal{A}^{(1)}_\nu \star \mathcal{A}^{(1)\nu} + \mathcal{B}^{(2)} \star \varkappa\rangle\,. \tag{5.22c}$$

5 This can be seen by observing that to linear order

$$\mathcal{S}_\alpha \star \mathcal{S}_\alpha = (z_\alpha + 2i\mathcal{A}_\alpha) \star (z_\alpha + 2i\mathcal{A}_\alpha) \approx z_\alpha \star z_\alpha + \{z_\alpha, 2i\mathcal{A}^{(1)}_\alpha\}_\star\,,$$

which vanishes for $z_\alpha = 0$ because $\mathcal{A}^{(1)}_\alpha$ is proportional to z_α and

$$\{z_\alpha, f(z,y)\}_\star = 2(z_\alpha + i\partial^y_\alpha)f(z,y)\,.$$

One can then obtain the second order physical and twisted equations of motion by inserting (5.22) in the dynamical equations of (5.10) which leads to

$$D^{yz}\mathcal{W}^{(2)}|_{z=0} = (\mathcal{W}^{(1)} \wedge \star \mathcal{W}^{(1)} - iE^{\alpha\alpha}\mathcal{A}^{(1)}_{\alpha} \star \mathcal{A}^{(1)}_{\alpha})|_{z=0}\,, \tag{5.23a}$$

$$D^{yz}\mathcal{B}^{(2)}|_{z=0} = [\mathcal{W}^{(1)}, \mathcal{B}^{(1)}]_{\star}|_{z=0}\,. \tag{5.23b}$$

To ease notation, all equations are implicitly understood to be evaluated at $z = 0$ in the following. Using the fact that we have set the first order twisted fields to zero, the above equations become after some algebra

$$\begin{aligned} D^{yz}\left(\hat{C}^{(2)}\psi + \tilde{C}^{(2)}\right) =& - D^{yz}\left(z^{\alpha}\Gamma_0\langle[\mathcal{A}^{(1)}_{\alpha}, \hat{C}^{(1)}\psi]_{\star}\rangle\right) \\ &+ [\hat{\omega}^{(1)} + M\psi, \hat{C}^{(1)}\psi]_{\star}\,, \end{aligned} \tag{5.24}$$

$$\begin{aligned} D^{yz}\left(\hat{\omega}^{(2)} + \tilde{\omega}^{(2)}\psi\right) =& D^{yz}\left(z^{\nu}\Gamma_0\langle[\bar{e}, \mathcal{A}^{(2)}_{\nu}]_{\star}\rangle\right) \\ &+ D^{yz}\left(z^{\nu}\Gamma_0\langle[\hat{\omega}^{(1)} + M\psi, \mathcal{A}^{(1)}_{\nu}]_{\star}\rangle\right) \\ &+ (\hat{\omega}^{(1)} + M\psi) \wedge \star(\hat{\omega}^{(1)} + M\psi) \\ &- iE^{\alpha\alpha}\mathcal{A}^{(1)}_{\alpha} \star \mathcal{A}^{(1)}_{\alpha}\,. \end{aligned} \tag{5.25}$$

Decomposing these equations in their physical and twisted components, we arrive at the following equations of motion:

$$(\tilde{D}\hat{C}^{(2)})\psi = \mathcal{V}(\hat{\omega}, \hat{C})\,, \tag{5.26a}$$

$$D\tilde{C}^{(2)} = \widetilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C})\,, \tag{5.26b}$$

$$(\tilde{D}\tilde{\omega}^{(2)})\psi = \widetilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)}) + \widetilde{\mathcal{V}}(\Omega, \hat{\omega}, \hat{C})\,, \tag{5.26c}$$

$$D\hat{\omega}^{(2)} = \mathcal{V}(\hat{\omega}, \hat{\omega}) + \mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})\,, \tag{5.26d}$$

with the source terms for the physical fields given by[6]

$$\begin{aligned} \mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C}) =& (M\psi) \wedge \star(M\psi) - iE^{\alpha\alpha}\mathcal{A}^{(1)}_{\alpha} \star \mathcal{A}^{(1)}_{\alpha} \\ &+ D^{yz}\left(z^{\nu}\Gamma_0\langle[M\psi, \mathcal{A}^{(1)}_{\nu}]_{\star}\rangle\right) \\ &+ D^{yz}\left(z^{\alpha}\Gamma_0\langle[\bar{e}, z_{\alpha}\Gamma_1\langle\mathcal{A}^{(1)}_{\nu} \star \mathcal{A}^{(1)\nu}\rangle]_{\star}\rangle\right) \\ &+ D^{yz}\left(z^{\alpha}\Gamma_0\langle[\bar{e}, z_{\alpha}\Gamma_1\langle\mathcal{B}^{(2)} \star \varkappa\rangle]_{\star}\rangle\right)\,, \end{aligned} \tag{5.27a}$$

$$\mathcal{V}(\hat{\omega}, \hat{\omega}) = \hat{\omega}^{(1)} \wedge \star\hat{\omega}^{(1)}\,, \tag{5.27b}$$

$$\mathcal{V}(\hat{\omega}, \hat{C}) = [\hat{\omega}^{(1)}, \hat{C}^{(1)}\psi]_{\star}\,, \tag{5.27c}$$

6 The field $\hat{C}^{(2)}$ does not contribute to the last term in $\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})$ by ψ-counting. It can also be shown that $\tilde{C}^{(2)}$ will not contribute to this term. Similarly, only $\hat{C}^{(2)}$ contributes to $\widetilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)})$.

and the source terms in the twisted sector read[7]

$$\begin{aligned}\widetilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C}) &= -D^{yz}\left(z^\nu \Gamma_0 \langle [\mathcal{A}_\nu^{(1)}, \hat{C}^{(1)}\psi]_\star\rangle\right) \\ &\quad + [M\psi, \hat{C}^{(1)}\psi]_\star \,, &(5.28a)\\ \widetilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)}) &= D^{yz}\left(z^\nu \Gamma_0 \langle [\bar{e}, \mathcal{A}_\nu^{(2)}]_\star\rangle\right) \,, &(5.28b)\\ \widetilde{\mathcal{V}}(\Omega, \hat{\omega}, \hat{C}) &= \{\hat{\omega}^{(1)}, M\psi\}_\star \\ &\quad + D^{yz}\left(z^\nu \Gamma_0 \langle [\hat{\omega}^{(1)}, \mathcal{A}_\nu^{(1)}]_\star\rangle\right) \,. &(5.28c)\end{aligned}$$

Obtaining an explicitly z_α-independent form of these source terms is a task of considerable technical difficulty and we will outline the main techniques we used in Section 5.4.2. The final form of the various source terms is given in Appendix B.1.

5.4.1 *Conservation Checks*

Nilpotency of the twisted adjoint and adjoint covariant derivative provides us with an important consistency check for our results: the source terms need to obey certain conservation identities. For example, for (5.26d), the relation $D^2 = 0$ implies

$$D\left(\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})\right) + D\hat{\omega}^{(1)} \wedge \star\hat{\omega}^{(1)} - \hat{\omega}^{(1)} \wedge \star D\hat{\omega}^{(1)} = 0 \,. \quad (5.29)$$

Using the first order equation of motion $D\hat{\omega}^{(1)} = 0$, this implies the following conservation law

$$D\left(\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})\right) = 0 \,. \quad (5.30)$$

Similar conservation laws can be derived for the other source terms. We carefully checked that our results are conserved. Especially for the source term $\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})$, this is a task of considerable technical difficulty which required the help of a computer algebra program.

5.4.2 *Explicit Evaluation of Source Terms*

In order to evaluate the source terms discussed in the last section, we have developed efficient methods which we will now illustrate for the example of

$$(M_2\psi) \wedge \star(M_1\psi)\big|_{z=0} \,, \quad (5.31)$$

which is part of the source term $\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})$ given by (5.27a) and M_1 and M_2 were defined in (3.123) and (3.116a) respectively. This term obviously contains two physical zero-forms $\hat{C}^{(1)}$ and it turns out to be efficient to consider their Fourier transformations for which we

7 From (5.28), it is not entirely obvious that the source terms are ψ-independent. However, it can be checked by an explicit calculation that this is indeed the case.

summarize the relevant conventions in Appendix A.4.3. In particular, we adopt the convention that the wave vector of the first $\hat{C}^{(1)}$ field is denoted by ξ_α and that of the second field (for the above example the zero-form in M_1) by η_α. This is important as each summand in the source term (5.27a) will lead to an expression of the form

$$\int \mathrm{d}^2\xi \mathrm{d}^2\eta\, f(y,\xi,\eta)\, \hat{C}^{(1)}(\xi,\phi|x)\psi\, \hat{C}^{(1)}(\eta,\phi|x)\psi \\ = \int \mathrm{d}^2\xi \mathrm{d}^2\eta\, f(y,\xi,\eta)\, \hat{C}^{(1)}(\xi,\phi|x)\, \hat{C}^{(1)}(\eta,-\phi|x)\,, \tag{5.32}$$

where we have used that ψ and ϕ anticommute (see (3.79)). This convention therefore amounts to associating a wave vector η_α with the zero-form that comes with a negative sign for ϕ. We now use the explicit form of M_2 and M_1 given in (3.123) and (3.116a) respectively. The free parameter in M_1 is set to $g_0 = 1$ for the sake of brevity. Using the integral representation of the star product (3.90), we then evaluate the source term (5.31) and obtain

$$-\frac{1}{32\pi^2}\bar{e}^{\alpha\alpha}\wedge\bar{e}^{\beta\beta}\left\{\int \mathrm{d}t\,\mathrm{d}q\,\mathrm{d}^2\xi\,\mathrm{d}^2\eta\,\mathrm{d}^2u\,\mathrm{d}^2v\;(1-t)(q^2-1)\right. \\ \times e^{iq(y+v)\eta-ity(\xi-u)+ivu}\,u_\alpha\left[(y+u)_\alpha(1-t)-(1+t)\xi_\alpha\right] \\ \left.\times(y+v-\eta)_\beta(y+v-\eta)_\beta\right\}\hat{C}^{(1)}(\xi,\phi|x)\,\hat{C}^{(1)}(\eta,-\phi|x)\,.$$

After shifting $u_\alpha \to u_\alpha - q\eta_\alpha$ and $v_\alpha \to v_\alpha - t(y+\xi)_\alpha$, the above expression becomes

$$-\frac{1}{32\pi^2}\bar{e}^{\alpha\alpha}\wedge\bar{e}^{\beta\beta}\left\{\int \mathrm{d}t\,\mathrm{d}q\,\mathrm{d}^2\xi\,\mathrm{d}^2\eta\,\mathrm{d}^2u\,\mathrm{d}^2v\;e^{ivu}\,R^2\,(1-t)(q^2-1)\right. \\ \times(u-q\eta)_\alpha[(y_\alpha+u_\alpha-q\eta_\alpha)(1-t)-(1+t)\xi_\alpha] \\ \left.\times(y_\beta+v_\beta-t(y+\xi)_\beta-\eta_\beta)(y_\beta+v_\beta-t(y+\xi)_\beta-\eta_\beta)\right\} \\ \times\hat{C}^{(1)}(\xi,\phi|x)\,\hat{C}^{(1)}(\eta,-\phi|x)\,,$$

where we have defined $R^2 := \exp[i\,q(y-t(\xi+y))\eta]$. We can now evaluate the integrals over u and v by using the following identities:

$$\frac{1}{(2\pi)^2}\int \mathrm{d}^2u\,\mathrm{d}^2v\;e^{ivu} = 1\,, \tag{5.33a}$$

$$\frac{1}{(2\pi)^2}\int \mathrm{d}^2u\,\mathrm{d}^2v\,e^{ivu}\;u_\alpha v_\beta = i\epsilon_{\alpha\beta}\,, \tag{5.33b}$$

whereas these types of integral vanish if the number u_α-oscillators is different from the number of v_α-oscillators. Using these identities, we arrive at our final result for (5.31) which reads

$$\int \mathrm{d}t\,\mathrm{d}q\,\mathrm{d}^2\xi\,\mathrm{d}^2\eta\;(q^2-1)\,R^2 \\ \times\left\{\frac{-i}{4}\,E^{\alpha\alpha}\left(T^2_\alpha T^1_\alpha + q(1-t)^2\eta_\alpha T^1_\alpha\right)\right. \\ \left.+\frac{1}{8}\,\bar{e}^{\alpha\alpha}\wedge\bar{e}^{\beta\beta}\;\left(q\;\eta_\alpha T^2_\alpha T^1_\beta T^1_\beta\right)\right\}\hat{C}^{(1)}(\xi,\phi|x)\,\hat{C}^{(1)}(\eta,-\phi|x)\,,$$

where the two-form $E^{\alpha\alpha}$ was defined in (3.119) and T^1 and T^2 are given by

$$T^1_\alpha := (1-t)y_\alpha - t\xi_\alpha - \eta_\alpha \,,$$
$$T^2_\alpha := (1-t^2)\xi_\alpha - (1-t)^2(y-q\eta)_\alpha \,.$$

In principle, one could try to further simplify this result by using the identity

$$\bar{e}^{\alpha\alpha} \wedge \bar{e}^{\beta\beta} = \frac{1}{4}\epsilon^{\alpha\beta}E^{\alpha\beta}\,, \tag{5.34}$$

but evaluating the resulting expression is quite cumbersome and is therefore most conveniently performed by using a computer algebra program.

5.5 TWISTED SECTOR RESULTS

In this section, we will discuss the equations of motion (5.26) for the twisted fields. We will start by analyzing the equations of motion for the zero-form $\tilde{C}^{(2)}$ before turning our attention towards the twisted one-form $\tilde{\omega}^{(2)}$. We will try to find a field frame which allows for vanishing solutions of the second order twisted fields - similar to our discussion of the linear order in Section 3.2.3.

5.5.1 *Twisted Zero-Form*

The equations of motion for the twisted zero-form were given in (5.26) and read

$$D\tilde{C}^{(2)} = \tilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C})\,. \tag{5.35}$$

Because the adjoint covariant derivative D commutes with the y_α-number operator $y^\nu\partial^y_\nu$, it is useful to use the following decomposition of $\hat{C}$

$$\hat{C}^{(2)} = \sum_{n=0}^{\infty} \hat{C}^{(2)}_{2n} \qquad \text{with} \qquad N\hat{C}^{(2)}_{2n} = 2n\,\hat{C}^{(2)}_{2n}\,, \tag{5.36}$$

as one can analyze (5.35) for each $\hat{C}^{(2)}_{2n}$ separately. We want to check whether there exists a field redefinition which removes the source term on the right-hand side of (5.35).

This question is of particular interest as the y_α-independent part of $\tilde{C}$ at zeroth order[8] specifies the λ-parameter of the $\mathfrak{hs}(\lambda)$ higher-spin theory as discussed in Section 3.2.2. If the y-independent component of the source term $\tilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C})$ cannot be removed by a field redefinition,

8 The interpretation of the y_α-independent component of $\tilde{C}$ at second and higher orders is less clear.

then the identity component of $\tilde{C}$ is necessarily deformed at second-order in perturbation theory.

We will now first show that the y_α-independent part of the source term can indeed be removed by a field redefinition. This proof then straightforwardly extends to y-dependent components of the source term.

The source term extracted from our second order analysis is of the form

$$\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C}) = \bar{e}^{\alpha\alpha}\int \mathrm{d}^2\xi\mathrm{d}^2\eta\, K_{\alpha\alpha}(\xi,\eta,y)\,\hat{C}^{(1)}(\xi,\phi)\,\hat{C}^{(1)}(\eta,-\phi)\,,$$

where the kernel $K_{\alpha\alpha}$ is given by

$$\begin{aligned}\phi\int_0^1 \mathrm{d}t\,\Big\{&\frac{1}{2}\,e^{i(y(1-t)-t\eta)\xi}\,\xi_\alpha\Big((1-t^2)\,(\xi_\alpha-\eta_\alpha)+(1-t)^2\,y_\alpha\Big)\\ &-\frac{1}{2}\,e^{i(y(1-t)-t\xi)\eta}\,\eta_\alpha\Big((1-t^2)\,(\eta_\alpha+\xi_\alpha)-(1-t)^2\,y_\alpha\Big)\\ &+\frac{1}{4}\,(t^2-1)\,e^{i(y-\eta)(y+t\xi)}\\ &\qquad\times\big(g_0\,(y-\eta)_\alpha(y-\eta)_\alpha-2(y-\eta)_\alpha\xi_\alpha+g_0\,\xi_\alpha\xi_\alpha\big)\\ &+\frac{1}{4}\,(t^2-1)\,e^{i(y+\xi)(t\eta-y)}\\ &\qquad\times\big(g_0\,(y+\xi)_\alpha(y+\xi)_\alpha-2\,(y-\xi)_\alpha\eta_\alpha+g_0\,\eta_\alpha\eta_\alpha\big)\Big\}\,.\end{aligned}$$

In order to check whether there is some field redefinition which removes the y-independent part of this source term, we consider $\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C})|_{y=0}$. Performing the t-integration in the corresponding kernel $K_{\alpha\alpha}(\xi,\eta,y=0)$ leads to

$$K_{\alpha\alpha}(\xi,\eta,y=0) = f(\eta\xi)\,\big(\,(1+g_0)\,\eta_\alpha\eta_\alpha-\,(1-g_0)\,\xi_\alpha\xi_\alpha\big)\,, \tag{5.37}$$

where we have defined $f(x)=4(x\cos(x)-x^{-3}\sin(x))$. We now want to cancel this contribution using a field redefinition

$$\tilde{C}^{(2)}\to\tilde{C}^{(2)}+\delta\tilde{C}^{(2)}(\hat{C}^{(1)},\hat{C}^{(1)})\,, \tag{5.38}$$

which will induce a contribution to the source term of the form

$$\delta\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C}) = -D\left(\delta\tilde{C}^{(2)}\right)\,. \tag{5.39}$$

As explained in Appendix A.4.3, the Fourier representation of the adjoint covariant derivative D is given by $\bar{e}^{\alpha\alpha}O_{\alpha\alpha}$ with

$$O_{\alpha\alpha} := \frac{i\phi}{2}\Big[(\eta_\alpha\eta_\alpha-\partial^\eta_\alpha\partial^\eta_\alpha)-\Big(\xi_\alpha\xi_\alpha-\partial^\xi_\alpha\partial^\xi_\alpha\Big)+2iy_\alpha\partial^y_\alpha\Big]\,. \tag{5.40}$$

As expected, $O_{\alpha\alpha}$ does not mix different powers of the y_α-oscillators. Therefore a field redefinition can change the y-independent part of the kernel $K_{\alpha\alpha}$ of $\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C})$ only by

$$\delta K_{\alpha\alpha}(\xi,\eta,y=0) = -O_{\alpha\alpha}F(\eta\xi) = \tfrac{i\phi}{2}\,(\xi_\alpha\xi_\alpha-\eta_\alpha\eta_\alpha)\,(F(\eta\xi)+F''(\eta\xi))\,,$$

where $F(x)$ is an arbitrary function. By comparison with (5.37), we therefore deduce that $\widetilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C})$ can only be canceled if

$$\boxed{g_0 = 0} \tag{5.41}$$

and one can easily derive a solution for $F(\eta\xi)$. Therefore, we can consistently set the y_α-independent component of $\tilde{C}^{(2)}$ to zero only for this choice of g_0. This procedure can be straightforwardly generalized to higher orders in y. One makes the most general ansatz for the function F which is now of the form

$$F(\xi\eta, \xi y, \eta y)\,, \tag{5.42}$$

but only contains up to a certain number of y_α-oscillators. Using $\delta K_{\alpha\alpha} = -O_{\alpha\alpha}F$, one then adjusts the free coefficients of the ansatz (5.42) such that the kernel $K_{\alpha\alpha}$ is canceled by the redefinition (up to the given order in y). Using a computer algebra program, this can be easily done and we found no obstructions in removing the source term $\widetilde{\mathcal{V}}(\Omega, \hat{C}, \hat{C})$ for the choice (5.41) - at least to the orders that were within numerical reach. In fact, we were able to prove in our publication [26] that the source term can be canceled to all orders in y_α-oscillators using a more involved formalism.

We also checked in [26] that this statement generalizes to the supersymmetric case. For this theory there is an additional free parameter d_0 in the source term and for only for $d_0 = 0$ the source term can be canceled.

5.5.2 *Twisted One-Form*

Following a similar strategy as in the last section, we will now construct a redefinition which removes the source terms in the equation of motion (5.26c) for the twisted one-form $\tilde{\omega}^{(2)}$ which we recall for convenience

$$(\tilde{D}\tilde{\omega}^{(2)})\psi = \tilde{\mathcal{V}}(\Omega, \hat{\omega}, \hat{C}) + \tilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)})\,. \tag{5.43}$$

The source term $\tilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)})$ is given by

$$\tilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)}) = \tfrac{1}{8}E^{\alpha\alpha}(y_\alpha + i\partial^u_\alpha)(y_\alpha + i\partial^u_\alpha)\hat{C}^{(2)}(u,\phi)\psi|_{u=0}\,, \tag{5.44}$$

and is therefore of the same form as the corresponding source term at linear order in (3.118b). Hence, we can remove it by performing a field redefinition $\tilde{\omega}^{(2)} \to \tilde{\omega}^{(2)} + M_1^{(2)}\psi$ with

$$M_1^{(2)} = \frac{\phi}{4}\,\bar{e}^{\alpha\alpha}\int_0^1 \mathrm{d}t\,(t^2-1) \\ \times (g_1\, y_\alpha y_\alpha + 2iy_\alpha t^{-1}\partial^y_\alpha - g_1\, t^{-2}\partial^y_\alpha\partial^y_\alpha)\hat{C}^{(2)}(ty)\,,$$

where g_1 is a free parameter. This can be shown as for the linear case (3.123), but now this field redefinition, apart from removing the source term $\tilde{\mathcal{V}}(\Omega, \Omega, \hat{C}^{(2)})$, also leads to an additional contribution to $\tilde{\mathcal{V}}(\Omega, \hat{\omega}, \hat{C})$ due to the fact that the equation of motion for $\hat{C}^{(2)}$ is given by $(\tilde{D}\hat{C}^{(2)})\psi = [\hat{\omega}^{(1)}, \hat{C}^{(1)}\psi]_\star$ as opposed to the linear case $\tilde{D}\hat{C}^{(1)} = 0$. Therefore, after performing this field redefinition, we obtain

$$(\tilde{D}\tilde{\omega}^{(2)})\psi = \tilde{\mathcal{V}}'(\Omega, \hat{\omega}, \hat{C})\,, \tag{5.45}$$

with a modified source term $\tilde{\mathcal{V}}'(\Omega, \hat{\omega}, \hat{C})$ which now depends on the parameter g_1 (and also g_0 of the redefinition (3.123) at linear order). This source term takes the following form in Fourier space

$$\tilde{\mathcal{V}}'(\Omega, \hat{\omega}, \hat{C}) = \bar{e}^{\alpha\alpha} \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta \left\{ L_{\alpha\alpha}(\xi, \eta, y)\, \hat{C}^{(1)}(\xi, \phi)\, \hat{\omega}^{(1)}(\eta, -\phi) \right. \\ \left. + \bar{L}_{\alpha\alpha}(\xi, \eta, y)\, \hat{\omega}^{(1)}(\xi, \phi)\, \hat{C}^{(1)}(\eta, \phi) \right\}, \tag{5.46}$$

The explicit expressions for the kernels $L_{\alpha\alpha}$ and $\bar{L}_{\alpha\alpha}$ are a bit lengthy and can be found in Appendix B.1.2.

We will now cancel this source term by a field redefinition

$$\tilde{\omega}^{(2)} \to \tilde{\omega}^{(2)} + \delta\tilde{\omega}^{(2)}(\hat{\omega}^{(1)}, \hat{C}^{(1)}) \tag{5.47}$$

which induces a change in the source term by

$$\delta\tilde{\mathcal{V}}'(\Omega, \hat{\omega}, \hat{C}) = -\tilde{D}\left(\delta\tilde{\omega}^{(2)}\right). \tag{5.48}$$

In Fourier space, the redefinition is given by

$$\delta\tilde{\omega}^{(2)} = \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta \left\{ F(\xi, \eta, y)\, \hat{C}^{(1)}(\xi, \phi)\, \hat{\omega}^{(1)}(\eta, -\phi) \right. \\ \left. + \bar{F}(\xi, \eta, y)\, \hat{\omega}^{(1)}(\xi, \phi)\, \hat{C}^{(1)}(\eta, \phi) \right\} \tag{5.49}$$

As is shown in Appendix A.4.3, the Fourier space representation of the twisted adjoint derivatives then leads to contributions

$$\delta L_{aa} = -I_{\alpha\alpha} F(y, \xi, \eta), \tag{5.50}$$
$$\delta \bar{L}_{aa} = -\bar{I}_{\alpha\alpha} \bar{F}(y, \xi, \eta), \tag{5.51}$$

in (5.46). The operator $I_{\alpha\alpha}$ by defined as

$$I_{\alpha\alpha} := \frac{i\phi}{2}\left[(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha) - \left(\xi_\alpha \xi_\alpha - \partial^\xi_\alpha \partial^\xi_\alpha\right) + 2i\, \eta_\alpha \partial^\eta_\alpha \right]. \tag{5.52}$$

It therefore preserves the degree in η_α-oscillators. Similarly, the contribution to $\bar{L}_{\alpha\alpha}$ preserves the number of ξ_α-oscillators. Such a field redefinition can therefore only induce the following contribution to the η-independent component of the $L_{\alpha\alpha}$ kernel

$$\delta L_{\alpha\alpha}(\xi, y, \eta = 0) = -I_{\alpha\alpha} F(\xi y) = \tfrac{i\phi}{2}\left(\xi_\alpha \xi_\alpha - y_\alpha y_\alpha\right)\left(F(\xi y) + F''(\xi y)\right).$$

This can be compared with the kernel $L_{\alpha\alpha}$ of the source term (5.46) which evaluated for vanishing η_α-oscillators gives

$$L_{\alpha\alpha}(\xi, y, \eta = 0) = \frac{1}{2}(g_0 - g_1)(y_\alpha y_\alpha + \xi_\alpha \xi_\alpha)\, f(\xi y), \tag{5.53}$$

where we defined $f(x) = 2\left(x \cos(x) - \sin(x)\right)/x^3$. As a result, the field redefinition cannot cancel this term and the only way out is to choose $g_1 = g_0$ which together with (5.41) implies

$$\boxed{g_1 = g_0 = 0}. \tag{5.54}$$

Using a completely analogous argument, one can check that the same condition is obtained by considering the ξ-independent part of $\bar{L}_{\alpha\alpha}$. It is also clear from our discussion that this argument can be straightforwardly extended to higher orders in η_α and ξ_α for unbarred and barred kernels respectively. For these higher orders, the source term will no longer vanish and can only be canceled if the condition (5.54) holds. In our publication [26], we also gave an all-order proof for this statement using more elaborate techniques.

In [26] it was furthermore checked that for the supersymmetric case the source term can only be canceled if one chooses $g_1 = g_0 = d_1 = d_0 = 0$ where d_0 and d_1 are additional free parameters appearing in the supersymmetric version of M_1 and $M_1^{(2)}$ respectively.

5.5.3 *Summary of Twisted Sector*

As we discussed in Section 3.2.3, at linear order it is possible to remove the source term of the twisted one-form but the necessary field redefinition M_1 contains a free parameter g_0. After this field redefinition, all twisted fields can be set to zero consistently.

Similarly, the source terms for the second order twisted fields can be removed by an appropriate field redefinitions. However, this is only possible for one particular choice of the parameter g_0, i.e.

$$g_0 = g_1 = 0\,, \tag{5.55}$$

where g_1 is a similar parameter arising at second order.

After removing the source terms by field redefinitions, one obtains the following second order equations of motion for the twisted fields

$$(\tilde{D}\tilde{\omega}^{(2)})\psi = 0\,, \qquad\qquad D\tilde{C}^{(2)} = 0\,, \tag{5.56}$$

and therefore one can consistently choose vanishing solutions for the second order twisted fields

$$\tilde{\omega}^{(2)} = 0\,, \qquad\qquad \tilde{C}^{(2)} = 0\,. \tag{5.57}$$

It seems then natural to conjecture that there exists a field frame in which one can choose vanishing solutions for the twisted fields to arbitrary order in perturbation theory and hence a truncation of the theory to physical fields only.

5.6 PHYSICAL SECTOR RESULTS

In this section, we will analyze the physical equations of motion to second order. We will determine them in the field frame of vanishing twisted fields and solve the (generalized) torsion constraint in order to make contact with the Fronsdal equation.

We will mainly consider the one-form sector because the second order equations of motion for the zero-form (5.26a) are rather trivial: these equations are just the straightforward generalizations of the free equations of motion $D_\Omega(\hat{C}^{(1)}\psi) = \mathrm{d}\hat{C}^{(1)}\psi - [\Omega, \hat{C}^{(1)}\psi]_\star = 0$ to the next order

$$\mathrm{d}\hat{C}\psi - [\Omega + \hat{\omega}, \hat{C}\psi]_\star = 0 \;\rightarrow\; (\tilde{D}\hat{C}^{(2)})\psi = [\hat{\omega}^{(1)}, \hat{C}^{(1)}\psi]_\star\,. \tag{5.58}$$

In [38, 55] the Gaberdiel–Gopakumar conjecture was successfully tested by comparing three-point functions involving two scalar fields and one higher-spin gauge field to the dual CFT 3-pt functions. These tests are however only performed using the second order equations of motion for the physical zero-form.

On the other hand, the second order equation of motion (5.26d) of the physical one-form $\hat{\omega}^{(2)}$ is given by

$$D\hat{\omega}^{(2)} = \hat{\omega}^{(1)} \wedge \star\, \hat{\omega}^{(1)} + \mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})\,. \tag{5.59}$$

The first term is expected from the Chern–Simons theory which describes the dynamics in the case of vanishing zero-form (and twisted one-form). The second term

$$J := \mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C}) \tag{5.60}$$

has no such interpretation. Due to the presence of this term, the gauge transformations of the higher-spin fields are deformed with respect to the ones of Chern–Simons theory. We will study this source term in detail in the following.

Note that the first term $\hat{\omega}^{(1)} \wedge \star\hat{\omega}^{(1)}$ leads to self-interactions of the higher-spin fields while the second term J contains the backreaction of the scalar fields to the higher-spin gauge fields. We will drop the first term in the following as we will study the backreaction of the scalar fields.

5.6.1 *Generalities*

The source term J of (5.60) is given in Fourier space by

$$J = E^{\alpha\alpha} \int \mathrm{d}^2\xi \mathrm{d}^2\eta\, K_{\alpha\alpha}(y, \xi, \eta)\, \hat{C}^{(1)}(\xi, \phi)\hat{C}^{(1)}(\eta, -\phi)\,, \tag{5.61}$$

with the kernel

$$\begin{aligned} K_{\alpha\alpha} &= y_\alpha y_\alpha\, f_1(\xi\eta, y\xi, y\eta) + y_\alpha \xi_\alpha\, f_2(\xi\eta, y\xi, y\eta) \\ &\quad + y_\alpha \eta_\alpha\, f_3(\xi\eta, y\xi, y\eta) + \xi_\alpha \xi_\alpha\, f_4(\xi\eta, y\xi, y\eta) \\ &\quad + \eta_\alpha \eta_\alpha\, f_5(\xi\eta, y\xi, y\eta) + \xi_\alpha \eta_\alpha\, f_6(\xi\eta, y\xi, y\eta)\,. \end{aligned} \tag{5.62}$$

The precise form of $f_{1\ldots 6}$ is given in Appendix B.1.1. Let us illustrate the interpretation of the various terms in (5.62) by considering a term in the kernel of the form

$$K_{\alpha\alpha} = \xi_{\alpha(N)} \eta_{\alpha(M)} y_{\alpha(2-N-M)}\, (y\xi)^n (y\eta)^m (\eta\xi)^l + \ldots\,, \tag{5.63}$$

where $N + M \leq 2$. By expanding the corresponding source term in the y_α-oscillators, i.e. $J = \sum_{k=0}^{\infty} \frac{1}{k!} J_{\alpha(k)} y^{\alpha(k)}$, one obtains the following tensor structure from this term

$$\begin{aligned} J_{\alpha(2+n+m-N-M)} \sim f^{n,m,l}_{N,M}\, & E^{\beta(N+M)}{}_{\alpha(2-N-M)} \\ & \times \hat{C}_{\beta(N)\alpha(n)\nu(l)}(\phi)\, \hat{C}^{\nu(l)}{}_{\beta(M)\alpha(m)}(-\phi) \\ & + \ldots\,. \end{aligned}$$

The constant $f^{n,m,l}_{N,M}$ is given by

$$f^{n,m,l}_{N,M} = (-1)^{N+M}(-i)^{N+M+n+m+2l}(m + n - N - M + 2)!\,. \tag{5.64}$$

Using (3.68), one can then rewrite $J_{\alpha(k)}$ in terms of derivatives of the physical scalar field Φ.

We can obviously decompose $J_{\alpha(k)}$ as follows

$$J_{\alpha(k)} = E^{\beta\beta}\,\mathsf{A}_{\alpha(k)\beta\beta} + E_{\alpha}{}^{\beta}\,\mathsf{B}_{\alpha(k-1)\beta} + E_{\alpha\alpha}\,\mathsf{C}_{\alpha(k-2)}\,, \tag{5.65}$$

where $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are zero-forms which are completely symmetric in all their spinorial indices. This observation will be important in Section 5.6.4.

5.6.2 *Independently Conserved Subsectors*

The adjoint covariant derivative D commutes with the y_α-number operator $y^\nu \partial^y_\nu$ and therefore each spin-component $J_{\alpha(k)}$ in the current J of (5.60) is conserved independently. However, as we will explain in the following, each spin-component splits even further into various independently conserved subsectors. To see this, it is again useful to consider the Fourier transformation of J. Let us define

$$\zeta^{\pm}_{\alpha} = (\xi \pm \eta)_\alpha\,. \tag{5.66}$$

In (5.62) the corresponding kernel $K_{\alpha\alpha}$ was parametrized by six functions $f_{1\ldots 6}$. We will now consider a different parameterization. Using the $\zeta^{\pm}_{\alpha}$, we can define the following contractions

$$Z_1 = \frac{1}{2} y\zeta^+\,, \qquad Z_2 = \frac{1}{2} y\zeta^-\,, \qquad Z_3 = \xi\eta\,, \tag{5.67}$$

and we can then decompose the kernel $K_{\alpha\alpha}$ as follows

$$K_{\alpha\alpha} = \sum_{n,m=0}^{\infty} \frac{1}{(n-1)!(m-1)!}\, K^{(n,m)}_{\alpha\alpha}\, Z_1^{n-1} Z_2^{m-1} \tag{5.68}$$

where we defined

$$\begin{aligned} K^{(n,m)}_{\alpha\alpha} &= y_\alpha y_\alpha\, k_1^{(n,m)}(Z_3) + y_\alpha \zeta^+_\alpha\, Z_2\, k_2^{(n,m)}(Z_3) \\ &\quad + y_\alpha \zeta^-_\alpha\, Z_1\, k_3^{(n,m)}(Z_3) + \zeta^+_\alpha \zeta^+_\alpha\, Z_2^2\, k_4^{(n,m)}(Z_3) \\ &\quad + \zeta^-_\alpha \zeta^-_\alpha\, Z_1^2\, k_5^{(n,m)}(Z_3) + \zeta^+_\alpha \zeta^-_\alpha\, Z_1 Z_2\, k_6^{(n,m)}(Z_3)\,. \end{aligned} \tag{5.69}$$

In the expression above, any negative power of Z_i is understood to be set to zero. This decomposition has the following nice property: each kernel $K^{(n,m)}_{\alpha\alpha}$ is independently conserved with respect to the adjoint covariant derivative D as was first shown in [56]. By counting powers in y, one observes that the spin of the kernel $K^{(n,m)}_{\alpha\alpha}$ is given by $2s = m + n + 2$ and therefore this decomposition splits each spin-component further in independently conserved pieces. For bosonic fields the kernel $K_{\alpha\alpha}$ is invariant under $\eta \to -\eta$. This symmetry exchanges Z_1 with Z_2 and therefore the sectors (n,m) and (m,n) are not independent for the bosonic truncation of the theory.

5.6.3 *Solving the Torsion Constraint*

In this section, we will discuss how the (generalized) torsion constraint can be solved in order to make contact with the Fronsdal equation in analogy to our discussion in Chapter 4.

To this end, we can decompose the covariant derivative as

$$D = \nabla + \phi Q \tag{5.70}$$

with $Q = -\bar{e}^{\alpha\alpha} y_\alpha \partial^y_\alpha$. The source term J of (5.60) can be split into $J = J^0 + \phi J^1$ with respect to ϕ and the second-order physical one-form decomposes as $\hat{\omega}^{(2)} = \omega^{(2)} + \phi e^{(2)}$. We can then rewrite the equation $D\hat{\omega}^{(2)} = J$ as

$$T'^{(2)} := \nabla e^{(2)} + Q\omega^{(2)} = J^1 \,, \tag{5.71a}$$
$$R'^{(2)} := \nabla \omega^{(2)} + Qe^{(2)} = J^0 \,, \tag{5.71b}$$

where we have dropped the term $\hat{\omega}^{(1)} \wedge \star\hat{\omega}^{(1)}$ on the right-hand side of (5.59) as discussed earlier. The explicit result for J^1 shows that the torsion $T'^{(2)}$ is non-vanishing.

We then solve the torsion constraint by first defining

$$\omega^{(2)} = \omega^{(2)}(e) + Q^{\#} J^1 \,, \tag{5.72}$$

where $Q^{\#}$ obeys $QQ^{\#}J = J$ for all Q-exact J.[9] The expression $Q^{\#}J^1$ is the contorsion one-form and $\omega^{(2)}(e)$ is the solution for $\omega^{(2)}$ in terms of vielbein e at vanishing torsion. Inserting this in (5.71) gives

$$T^{(2)} := \nabla e^{(2)} + Q\omega^{(2)}(e) = 0 \,, \tag{5.73a}$$
$$R^{(2)} := \nabla \omega^{(2)}(e) + Qe^{(2)} = j \,, \tag{5.73b}$$

where j is given by

$$j = J^0 - \nabla Q^{\#} J^1 \,. \tag{5.74}$$

It is important to note that the operator $Q^{\#}$ is well-defined and in the basis (5.65) reads[10]

$$(Q^{\#}J)_{\alpha(k)} = \frac{2}{k}\bar{e}^{\beta\beta}\mathsf{A}_{\alpha(k)\beta\beta} - \bar{e}_\alpha{}^\beta \mathsf{B}_{\alpha(k-1)\beta} - \frac{2}{k+2}\bar{e}_{\alpha\alpha}\mathsf{C}_{\alpha(k-2)} \,. \tag{5.75}$$

In the following section, we will study j more closely and discuss how it is related to the Fronsdal current.

---

9 We are implicitly using here the analysis of the σ_--cohomology of which some aspects are discussed in Appendix C.

10 $k > 0$ is implied in this relation as there is no torsion constraint to be solved for the case of spin 1.

5.6.4 *Obtaining the Fronsdal Current*

At the second order, the Fronsdal equation acquires a source term

$$F_{m(s)} := \Box\phi_{m(s)} + \cdots = \mathsf{j}_{m(s)} \,. \tag{5.76}$$

We will refer to $\mathsf{j}_{m(s)}$ as the *Fronsdal current*. In spinorial language the Fronsdal operator $F_{m(s)}$ decomposes into two components, $F_{\alpha(2s)}$ and $F_{\alpha(2s-4)}$, which respectively correspond to its traceless and trace part. Similarly, the Fronsdal current decomposes into $\mathsf{j}_{\alpha(2s)}$ and $\mathsf{j}'_{\alpha(2s-4)}$.

We will now show that these components of the Fronsdal operator and current can be extracted from the curvature $R^{(2)}$ and the current j respectively.

Let us first note that nilpotency of D and conservation of J imply the following relations:

$$D^2 = 0 \quad \rightarrow \quad \{\nabla, Q\} = 0\,, \qquad \nabla^2 + Q^2 = 0\,, \tag{5.77a}$$

$$DJ = 0 \quad \rightarrow \quad \nabla J^0 + QJ^1 = 0\,, \quad QJ^0 + \nabla J^1 = 0\,. \tag{5.77b}$$

Using these relations one derives

$$\nabla j = \nabla R^{(2)} = 0\,, \qquad Qj = QR^{(2)} = 0\,. \tag{5.78}$$

These equations correspond to the differential and algebraic Bianchi identities respectively. The first condition implies that the j of (5.74) is conserved with respect to the Lorentz covariant derivative ∇. The second condition implies that

$$\bar{e}_\alpha{}^\beta \wedge j_{\beta\alpha(k-1)} = \bar{e}_\alpha{}^\beta \wedge R^{(2)}_{\beta\alpha(k-1)} \equiv 0\,. \tag{5.79}$$

By using (A.18b), one can show that this is only guaranteed to hold if and only if $\mathsf{B} \equiv 0$ in the decomposition (5.65) for $R^{(2)}_{\alpha(2s)}$ and $j_{\alpha(2s)}$. Therefore, we decompose

$$R^{(2)}_{\alpha(2s-2)} = j_{\alpha(2s-2)} \tag{5.80}$$

by using

$$R^{(2)}_{\alpha(2s-2)} = E^{\beta\beta}\, F_{\alpha(2s-2)\beta\beta} + E_{\alpha\alpha}\, F'_{\alpha(2s-4)}\,, \tag{5.81}$$

$$j_{\alpha(2s-2)} = E^{\beta\beta}\, \mathsf{j}_{\alpha(2s-2)\beta\beta} + E_{\alpha\alpha}\, \mathsf{j}'_{\alpha(2s-4)}\,. \tag{5.82}$$

By index counting, it then follows that $\mathsf{j}_{\alpha(2s)}$ and $\mathsf{j}'_{\alpha(2s-4)}$ correspond to the traceless and trace component of the Fronsdal current $\mathsf{j}_{n(s)}$ of (5.76). Similarly, $F_{\alpha(2s)}$ and $F'_{\alpha(2s-4)}$ encode the trace and traceless part of the Fronsdal operator. Due to this identification, we will refer to j of (5.74) also as Fronsdal current in the following. The components in (5.82) can be conveniently expressed by

$$\mathsf{j}_{\alpha(2s+2)} = \sum_l \sum_{n+m=2s} a^{n,m,l}\, \hat{C}_{\alpha(n+1)\beta(l)}(\phi)\, \hat{C}^{\beta(l)}{}_{\alpha(m+1)}(-\phi)\,,$$

$$\mathsf{j}'_{\alpha(2s-2)} = \sum_l \sum_{n+m=2s} c^{n,m,l}\, \hat{C}_{\alpha(n-1)\beta(l)}(\phi)\, \hat{C}^{\beta(l)}{}_{\alpha(m-1)}(-\phi)\,.$$

Before summarizing our explicit results for the coefficients $a^{n,m,l}$ and $c^{n,m,l}$ of j, we will first discuss what is known about their form from general arguments.

5.6.5 *Expectation for the Result*

It was shown in [57] that at second order the corrections to the Fronsdal equation bilinear in the scalar fields can be fixed up to a spin-dependent constant c_s and field redefinitions:

$$F_{m(s)} = c_s \, \mathrm{j}^{\min}_{m(s)}(\Phi^\dagger, \Phi) \,, \tag{5.84}$$

where $\mathrm{j}^{\min}_{m(s)}(\Phi^\dagger, \Phi)$ contains only up to s derivatives and can be chosen to be completely traceless (see also [**Bekaert:2010hk**]). We will refer to this source term as the spin-s *minimal current* in the following.

The minimal current corresponds to[11]

$$j^s(y) = \hat{C}^{(1)}(y, \phi) \, \hat{C}^{(1)}(y, -\phi)\Big|_{y^{\alpha(2s)}} \tag{5.85}$$

in the language of Vasiliev theory. Here $\bullet|_{y^{\alpha(2s)}}$ denotes all terms involving $2s$ oscillators y_α. By (3.63), this expression contains only up to s derivatives of the physical scalar field Φ. The expression (5.85) is related to the minimal current by[12]

$$j^s(y) \sim \frac{1}{(2s)!} \mathrm{j}^{\min}_{\alpha(2s)} \, y^{\alpha(2s)} \,. \tag{5.86}$$

From our discussion in the last section and the fact that the minimal current is completely traceless, we expect the following second order relations

$$R^{(2)}_{\alpha(2s)} = E^{\beta\beta} F_{\beta\beta\alpha(2s)} + E_{\alpha\alpha} F_{\alpha(2s-2)} = \tilde{c}^s \, E^{\beta\beta} \mathrm{j}^{\min}_{\beta\beta\alpha(2s)} \,. \tag{5.87}$$

We will now show that this also implies that the spin-s minimal current only resides in $k_4^{(2s-1,-1)}$ component of the $(2s-1,-1)$ sector of the backreaction in the notation of Section 5.6.2 (and therefore also in the $k_5^{(-1,2s-1)}$ component of the $(-1, 2s-1)$ sector for the bosonic case).

To show this, we contract the two-form on the right hand side of the relation above with y_α oscillators

$$\frac{1}{2s!} \tilde{c}^s \, E^{\beta\beta} \mathrm{j}^{\min}_{\beta\beta\alpha(2s)} y^{\alpha(2s)} = \tilde{c}^s E^{\alpha\alpha} \partial_\alpha \partial_\alpha \, j^s(y) \,. \tag{5.88}$$

This expression has the following kernel $K_{\alpha\alpha}$ in Fourier space

$$K_{\alpha\alpha} \sim \tilde{c}^s \, \zeta^+_\alpha \zeta^+_\alpha \, (Z_1)^{2s-2} \,, \tag{5.89}$$

---

11 The relative sign in ϕ ensures that the current is of the form $\Phi^\dagger \dots \Phi$.

12 Here we use the notation $y^{\alpha(2s)} = y^{\alpha_1} \dots y^{\alpha_{2s}}$.

with $Z_1 = \frac{1}{2}\zeta^+ y$ and $\zeta^+_\alpha = (\xi+\eta)_\alpha$. This can be seen by the fact that j^s is given in Fourier space by

$$j^s(y) = \int \mathrm{d}^2\xi\, \mathrm{d}^2\xi\, e^{iy(\xi+\eta)}\, \hat{C}^{(1)}(\xi,\phi)\, \hat{C}^{(1)}(\eta,-\phi)\Big|_{y^{\alpha(2s)}}$$
$$\sim \int \mathrm{d}^2\xi\, \mathrm{d}^2\xi\, (Z_1)^{2s}\, \hat{C}^{(1)}(\xi,\phi)\, \hat{C}^{(1)}(\eta,-\phi)\,.$$

By inspecting the explicit form of $K_{\alpha\alpha}$, it is clear from our discussion in Section 5.6.2 that the spin-s minimal current indeed only contributes to the $k_4^{(2s-1,-1)}$ component.

We would like to extract the coefficients $\tilde{c}^s$ from the explicit results discussed in the next section.

5.6.6 *Explicit Results*

The explicit results for arbitrary spins, which we obtain from Vasiliev theory, are rather involved. In the following, we will therefore only discuss the source term of the spin-2 gauge fields.

From our discussion in Section 5.6.2, it follows that we have five independently conserved subsectors $(3,-1)$, $(2,0)$, $(1,1)$, $(0,2)$, $(-1,3)$ in the case of spin-2. However, we are considering bosonic fields and therefore the sectors (n,m) and (m,n) are not independent as also discussed in Section 5.6.2. Thus the backreaction of the scalar fields splits into three separately conserved components for spin-2:

$$R^{(2)}_{\alpha\alpha} = j_{\alpha\alpha} = j^{(3,-1)}_{\alpha\alpha} + j^{(1,1)}_{\alpha\alpha} + j^{(2,0)}_{\alpha\alpha}\,. \tag{5.90}$$

We find the following expressions for these components

$$j^{(3,-1)}_{\alpha\alpha} = E^{\beta\beta}\, \mathsf{j}^{(3,-1)}_{\alpha\alpha\beta\beta}\,, \tag{5.91a}$$
$$j^{(1,1)}_{\alpha\alpha} = E^{\beta\beta}\, \mathsf{j}^{(1,1)}_{\alpha\alpha\beta\beta} + E_{\alpha\alpha}\, \mathsf{j}'^{(1,1)}\,, \tag{5.91b}$$
$$j^{(2,0)}_{\alpha\alpha} \equiv 0\,. \tag{5.91c}$$

The source terms on the right hand side of this equation are given by

$$\mathsf{j}^{(3,-1)}_{\alpha(4)} = \sum_{l\in 2\mathbb{N}} a_l \Big(\hat{C}_{\alpha(4)\nu(l)}(\phi)\, \hat{C}^{\nu(l)}(-\phi) + 3\, \hat{C}_{\alpha(2)\nu(l)}(\phi)\, \hat{C}^{\nu(l)}{}_{\alpha(2)}(-\phi)\Big)\,, \tag{5.92a}$$
$$\mathsf{j}^{(1,1)}_{\alpha(4)} = \sum_{l\in 2\mathbb{N}} b_l \Big(\hat{C}_{\alpha(4)\nu(l)}(\phi) \hat{C}^{\nu(l)}(-\phi) - \hat{C}_{\alpha(2)\nu(l)}(\phi)\, \hat{C}^{\nu(l)}{}_{\alpha(2)}(-\phi)\Big)\,, \tag{5.92b}$$
$$\mathsf{j}'^{(1,1)} = \sum_{l\in 2\mathbb{N}} b'_l\, \hat{C}_{\nu(l)}(\phi)\, \hat{C}^{\nu(l)}(-\phi)\,, \tag{5.92c}$$

where projection on the ϕ-independent part is implied. The coefficients read as follows

$$
\begin{aligned}
a_l &= \frac{i^{l-1}}{4l!}\left(\frac{1}{1+l}-\frac{6}{2+l}+\frac{9}{(3+l)^2}+\frac{19}{4(3+l)}-\frac{6}{4+l}+\frac{7}{5+l}-\frac{3}{4(7+l)}\right)\,,\\
b_l &= -\frac{i^{l-1}}{4l!}\left(\frac{1}{2+l}-\frac{1}{(3+l)^2}-\frac{13}{4\,(3+l)}+\frac{4}{4+l}-\frac{1}{5+l}-\frac{1}{6+l}+\frac{1}{4(7+l)}\right)\,,\\
b'_l &= \frac{i^{l-1}}{l!}\left(\frac{1}{3(1+l)^2}+\frac{7}{12(1+l)}-\frac{3}{2+l}+\frac{1}{3+l}+\frac{1}{3(4+l)}-\frac{1}{4(5+l)}-\frac{1}{6}\delta_{l,0}\right)\,.
\end{aligned}
$$

We will discuss the relation of these results to the expected minimal current in the next section.

5.6.7 *Relating Results to Minimal Current*

We have seen in Section 5.6.5 that for spin-2 the minimal current resides in the $(3,-1)$ subsector. This corresponds to $j^{(3,-1)}_{\alpha\alpha}$ in the notation of the last section. All other contributions to the spin-2 source term should therefore be removable by field redefinitions. However, the terms in the other independently conserved (and non-vanishing) subsector $j^{(1,1)}_{\alpha\alpha}$ contain infinitely many derivatives of the scalar field Φ at fixed spin. This can be seen from the explicit results for $j^{(1,1)}_{\alpha\alpha}$ given by (5.92b) and (5.92c) combined with

$$
\hat{C}_{\alpha(2s)} \sim \left(\nabla_{\alpha\alpha}\right)^s \Phi\,. \tag{5.93}
$$

In order to remove this sector, one needs a field redefinition of $\hat{\omega}^{(2)}_{\alpha\alpha}$ by an expression of the form

$$
\sum_{k=0}^{2}\sum_{l=0}^{\infty} f_{k,l}\,\hat{C}_{\alpha(2-k)\nu(l)}\,\hat{C}_{\alpha(k)}{}^{\nu(l)}\,. \tag{5.94}
$$

where one has to appropriately contract with $\bar{e}^{\alpha\alpha}$ to obtain an expressions of form-degree one. By (5.93), this field redefinition contains generically an infinite number of derivatives and is therefore potentially nonlocal. It has been shown in [56] that by field redefinitions of this form one can even remove the spin-2 minimal current. This suggests that some of the field redefinitions (5.94) are not physically allowed.

Let us discuss this phenomenon in more general terms. We define a *pseudo-local field redefinition* to be of the form

$$
\sum_{k=0}^{s}\sum_{l=0}^{\infty} f_{s,k,l}\,\hat{C}_{\alpha(s-k)\nu(l)}\,\hat{C}_{\alpha(k)}{}^{\nu(l)}\,, \tag{5.95}
$$

where one has to appropriately contract with $E^{\alpha\alpha}$ or $\bar{e}^{\alpha\alpha}$ for redefinitions of form-degree 1 and 2 respectively. In our publication [26] we extended the proof in [56] by showing that pseudo-local field redefinitions can remove any J on the right hand side of the physical one-form equation of motion, i.e.

$$
D\hat{\omega}^{(2)} = J(\hat{C}^{(1)},\hat{C}^{(1)}) \quad \xrightarrow[(5.95)]{\text{redefinition of type}} \quad D\hat{\omega}^{(2)} = 0\,. \tag{5.96}
$$

This confronts us with a serious challenge: on the one hand, pseudo-local field redefinitions are required to make contact with the minimal current and extract its coefficients. On the other hand, by such field redefinitions, one can remove any interaction term and in particular the minimal current. It also follows that we can arbitrarily change the coefficient in front of the minimal current (as we can choose to remove a certain multiple of it). This suggests that some pseudo-local field redefinitions are physically not permitted and therefore a criterion for the subset of physically allowed pseudo-local field redefinitions has to be found.

The field redefinition (5.38) removing the source term to the twisted zero-form is also of pseudo-local type. However, this may not necessarily represent a problem even if the required field redefinition was not contained within the physically allowed class. Because in this case, one might simply define the physical theory by this field frame.

At this stage, one might wonder if the fact that we can remove all interactions is a pathology of the three-dimensional theory. After all (higher-spin) gauge fields do not propagate in three dimensions and so one has to be careful with what is meant by an interacting theory as the gauge degrees of freedom will only reside at the boundary. In the next chapter, we will turn our attention towards the four-dimensional Vasiliev theory and repeat our analysis for this case. We will however see that the problem is also present in four dimensions and we will therefore return to the question of classifying allowed pseudo-local field redefinitions afterwards.

Part III

FOUR DIMENSIONS

6 FOUR DIMENSIONAL VASILIEV THEORY

6.1 SPINORIAL DICTIONARY

As we have seen in the last chapters, three-dimensional Vasiliev theory is formulated in spinorial language. In four dimensions this is also the case. It is therefore important to establish a dictionary between Lorentz tensors and multi-spinors in four dimensions.

To this end we define

$$\sigma^a_{\alpha\dot{\alpha}} = (\mathbb{1}, \sigma^i)\,, \tag{6.1}$$

where $\alpha, \dot{\alpha} \in \{0,1\}$ and σ^i denote the three Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{6.2}$$

It is also convenient to define

$$\bar{\sigma}^{a\,\dot{\alpha}\alpha} := \epsilon^{\alpha\beta}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\sigma^a_{\beta\dot{\beta}}\,, \tag{6.3}$$

where we introduced $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ with $\epsilon^{01} = 1$ which can be used to lower and raise spinorial indices

$$v^\alpha = \epsilon^{\alpha\beta} v_\beta\,, \qquad\qquad v_\alpha = v^\beta \epsilon_{\beta\alpha}\,, \tag{6.4}$$

and analogously for the dotted indices. In components it follows that $\bar{\sigma}^a = (\mathbb{1}, -\sigma^i)$ and that

$$(\sigma^a_{\alpha\dot{\alpha}})^* = \bar{\sigma}^a_{\dot{\alpha}\alpha}\,, \tag{6.5}$$

where the right hand side is obtained by $\bar{\sigma}^a_{\dot{\alpha}\alpha} = \bar{\sigma}^{a\,\dot{\beta}\beta}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\beta\alpha}$ in accordance with (6.4). It is also natural to introduce

$$(\sigma^{ab})_\alpha{}^\beta := -\frac{1}{2}\left(\sigma^a_{\alpha\dot{\gamma}}\,\bar{\sigma}^{b\dot{\gamma}\beta} - \sigma^b_{\alpha\dot{\gamma}}\,\bar{\sigma}^{a\dot{\gamma}\beta}\right)\,, \tag{6.6}$$

$$(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} := -\frac{1}{2}\left(\bar{\sigma}^{a\,\dot{\alpha}\gamma}\,\sigma^b_{\gamma\dot{\beta}} - \bar{\sigma}^{b\,\dot{\alpha}\gamma}\,\sigma^a_{\gamma\dot{\beta}}\right)\,. \tag{6.7}$$

From (6.5), one immediately concludes that $[(\sigma^{ab})_\alpha{}^\beta]^* = (\bar{\sigma}^{ab})^{\dot{\beta}}{}_{\dot{\alpha}}$. It is also important to note that $(\sigma^{ab})_{\alpha\beta}$ and $(\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}$ are symmetric in their spinorial indices but antisymmetric in their Lorentz indices.

Using these definitions, one can derive a dictionary between Lorentz tensors and their spinorial counterparts. For example a Lorentz vector v^a is mapped to

$$v_{\alpha\dot{\alpha}} := v_a \sigma^a_{\alpha\dot{\alpha}}\,. \tag{6.8}$$

The map (6.8) originates from the fact that the group $SL(2,\mathbb{C})$ is the double cover of $SO^+(1,3)$: it can be easily checked that $v^a v_a = det(v_a\sigma^a)$. Elements of $SL(2,\mathbb{C})$ act on (6.8) by conjugation and preserve the determinant. Therefore, there exists a group homomorphism $\Phi : SL(2,\mathbb{C}) \to SO^+(1,3)$. Its kernel consists of those matrices which induce trivial transformations, i.e. $\Phi^{-1}(\mathbb{1}_{4x4}) = \{\mathbb{1}_{2x2}, -\mathbb{1}_{2x2}\}$.

Using the identity $-\frac{1}{2}\sigma^a_{\alpha\dot\alpha}\bar\sigma_b^{\dot\alpha\alpha} = \delta^a_b$ it follows that

$$v^a = -\frac{1}{2} v_{\alpha\dot\alpha}\bar\sigma^{a\,\dot\alpha\alpha} \,. \tag{6.9}$$

To illustrate this map further, let us consider an antisymmetric Lorentz tensor $C^{a,b} = -C^{b,a}$ which in spinorial language becomes

$$\begin{aligned} C^{a,b}\sigma_{a\,\alpha\dot\alpha}\,\sigma_{b\,\beta\dot\beta} &= C^{a,b}\left(-\frac{1}{2}\,\epsilon_{\alpha\beta}\,(\bar\sigma_{ab})_{\dot\alpha\dot\beta} - \frac{1}{2}\,\epsilon_{\dot\alpha\dot\beta}\,(\sigma_{ab})_{\alpha\beta}\right) \\ &= C_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} + \bar C_{\dot\alpha\dot\beta}\,\epsilon_{\alpha\beta}\,, \end{aligned} \tag{6.10}$$

where we have used that the completely symmetric and antisymmetric part in $\alpha \leftrightarrow \beta$ and $\dot\alpha \leftrightarrow \dot\beta$ will not contribute by symmetry. Furthermore, we defined the symmetric tensors

$$C^{\alpha\beta} := -\frac{1}{2}\,C^{a,b}\,(\sigma_{ab})^{\alpha\beta}\,, \qquad \bar C^{\dot\alpha\dot\beta} := -\frac{1}{2}\,C^{a,b}\,(\bar\sigma_{ab})^{\dot\alpha\dot\beta}\,. \tag{6.11}$$

For real $C^{a,b}$, it then follows immediately that $(C^{\alpha\beta})^* = \bar C^{\dot\alpha\dot\beta}$. Using the formulae

$$(\sigma^{ab})^{\alpha\beta}\,(\sigma_{cd})_{\alpha\beta} = 2(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) - 2i\epsilon^{ab}{}_{cd}\,, \tag{6.12}$$

$$(\bar\sigma^{ab})^{\dot\alpha\dot\beta}\,(\bar\sigma_{cd})_{\dot\alpha\dot\beta} = 2(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) + 2i\epsilon^{ab}{}_{cd}\,, \tag{6.13}$$

one can invert these relations

$$C^{a,b} = -\frac{1}{4}\left(C^{\alpha\beta}(\sigma^{ab})_{\alpha\beta} + \bar C^{\dot\alpha\dot\beta}(\bar\sigma^{ab})_{\dot\alpha\dot\beta}\right)\,.$$

We have therefore shown that an antisymmetric Lorentz tensor $C^{a,b}$ maps to

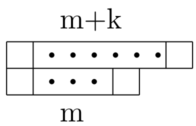
$$C^{a,b} \qquad \longleftrightarrow \qquad C_{\alpha\beta}\oplus\bar C_{\dot\alpha\dot\beta}\,, \tag{6.14}$$

where $C_{\alpha\beta}$ and $\bar C_{\dot\alpha\dot\beta}$ are related by complex conjugation. A real antisymmetric tensor $C^{a,b}$ in four dimensions has six real degrees of freedom which matches the six real degrees of freedom encoded in the complex symmetric 2×2-matrix $C_{\alpha\beta}$ (and its complex conjugate $\bar C_{\dot\alpha\dot\beta}$).

This statement can be readily generalized: consider the following complex conjugated pair of multispinors

$$T^{\alpha(k+2m),\dot\alpha(k)}\,, \qquad \bar T^{\alpha(k),\dot\alpha(k+2m)}\,. \tag{6.15}$$

This pair corresponds to a Lorentz tensor $T^{a(m+k),b(m)}$ transforming in a representation of the Lorentz algebra given by the following Young diagram:



This can be roughly understood as follows: consider for example $T^{\alpha(k+2m),\dot\alpha(k)}$. We can contract k dotted and undotted indices with $\sigma^a_{\alpha\dot\alpha}$

and the remaining $2m$ undotted indices with $\sigma^{ab}_{\alpha\alpha}$. We proceed similarly for $T^{\alpha(k),\dot{\alpha}(k+2m)}$ and add the results. Using the symmetry properties of $\sigma^{a}_{\alpha\dot{\alpha}}$, $\sigma^{ab}_{\alpha\alpha}$ and $\bar{\sigma}^{ab}_{\dot{\alpha}\dot{\alpha}}$, one can then indeed show that the resulting tensor has symmetry properties corresponding to the Young diagram above.

As we discussed in Chapter 2, the frame-like formulation for free higher-spin gauge fields on an AdS$_4$-background is given in terms of the fields $\omega^{a(s-1),b(t)}$ with $0 \leq t \leq s-1$ (see (2.49)). These fields are equivalent to

$$\omega^{\alpha(s-1+t),\dot{\alpha}(s-1-t)}\,, \qquad\qquad \omega^{\alpha(s-1-t),\dot{\alpha}(s-1+t)}\,, \tag{6.16}$$

which are related by complex conjugation. In particular this implies that the generalized vielbein, $\omega^{a(s-1)} = e^{a(s-1)}$, corresponds to the self-conjugated field $\omega^{\alpha(s-1),\dot{\alpha}(s-1)}$. We will rewrite the linearized curvatures (2.49) in the spinorial formalism as discussed in the following. For this, it is however first necessary to define an AdS$_4$ background in this formalism.

6.2 ADS$_4$ BACKGROUND

Using the map discussed in the last section, we can rewrite the AdS$_4$ isometry algebra (3.10) in terms of $P_{\alpha\dot{\alpha}}$, $L_{\alpha\alpha}$ and $L_{\dot{\alpha}\dot{\alpha}}$. Using (6.8) and (6.10), it then becomes

$$[P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = \frac{1}{l^2}\left(L_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}} + L_{\dot{\alpha}\dot{\beta}}\,\epsilon_{\alpha\beta}\right)\,, \tag{6.17a}$$

$$[L_{\alpha\alpha}, P_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} P_{\alpha\dot{\beta}}\,, \qquad [L_{\dot{\alpha}\dot{\alpha}}, P_{\beta\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} P_{\beta\dot{\alpha}}\,, \tag{6.17b}$$

$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}\,, \qquad [L_{\dot{\alpha}\dot{\alpha}}, L_{\dot{\beta}\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} L_{\dot{\alpha}\dot{\beta}}\,. \tag{6.17c}$$

We now proceed in analogy to the three-dimensional case by introducing a pair of commuting oscillators y_α and $\bar{y}_{\dot{\alpha}}$. It is convenient to collect them into

$$Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})\,. \tag{6.18}$$

Indices of Y^A can then be lowered or raised by

$$\epsilon^{AB} := \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{6.19}$$

using analogous conventions to (6.4). These oscillators can be multiplied by the following star product

$$(f \star g)(Y) = \frac{1}{(4\pi)^4} \int \mathrm{d}^4U \mathrm{d}^4V\, f(Y+U) g(Y+V)\, e^{iVU}\,, \tag{6.20}$$

where we have defined $VU := V^A U_A$. Note that for purely y or $\bar{y}$ dependent functions, this reduces to the three-dimensional (undeformed) star product (3.27). From these definitions, it then follows after some algebra that

$$L_{\alpha\alpha} = -\frac{i}{2} y_\alpha y_\alpha \,, \qquad L_{\dot\alpha\dot\alpha} = -\frac{i}{2} \bar y_{\dot\alpha} \bar y_{\dot\alpha} \,, \qquad P_{\alpha\dot\alpha} = -\frac{i}{2l} y_\alpha \bar y_{\dot\alpha} \,, \tag{6.21}$$

obey (6.17) after replacing the commutators by star commutators. This statement can be shown along very similar lines as our discussion of the three-dimensional case in Section 3.1.2. The main difference between three and four dimensions is that we are not forced to introduce any additional Clifford elements (see (3.35)) to realize the local translation generator P_a due to the presence of $\bar y_{\dot\alpha}$ oscillators in addition to y_α.

In Section 3.1.1, it was shown that, in any dimension, maximally symmetric spacetimes can be described by a one-form Ω obeying the zero-curvature condition (3.8). We then obtain the following connection for an AdS$_4$ background

$$\Omega = \frac{1}{2} \bar\omega^{\alpha\alpha} L_{\alpha\alpha} + \frac{1}{2} \bar\omega^{\dot\alpha\dot\alpha} L_{\dot\alpha\dot\alpha} + \bar e^{\alpha\dot\alpha} P_{\alpha\dot\alpha} \,, \tag{6.22}$$

obeying the equation of motion

$$\mathrm{d}\Omega = \Omega \wedge \star \Omega \,. \tag{6.23}$$

We normalize the background vielbein such that it obeys the following identities

$$\bar e_n^{\alpha\dot\alpha} \bar e^n_{\beta\dot\beta} = -\frac{1}{2} \delta^\alpha_\beta \, \delta^{\dot\alpha}_{\dot\beta} \,, \qquad\qquad \bar e_m^{\alpha\dot\alpha} \bar e^n_{\alpha\dot\alpha} = -\frac{1}{2} \delta^n_m \,. \tag{6.24}$$

One also introduces an AdS$_4$-covariant derivative for a differential form F of degree $|F|$

$$\begin{aligned} D_\Omega F :=& \mathrm{d}F - \Omega \wedge \star F + (-1)^{|F|} F \wedge \star \Omega \\ =& \nabla F - \frac{1}{l} \bar e^{\alpha\dot\alpha} \left(y_\alpha \partial_{\dot\alpha} + \bar y_{\dot\alpha} \partial_\alpha \right) F \,, \end{aligned} \tag{6.25}$$

where we have defined the Lorentz covariant derivative

$$\nabla \bullet = (\mathrm{d} - \bar\omega^{\alpha\beta} y_\alpha \partial_\beta - \bar\omega^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \partial_{\dot\beta}) \bullet \,. \tag{6.26}$$

Since the background connection Ω obeys the zero curvature condition (6.23), it follows that D_Ω is nilpotent

$$D_\Omega^2 = 0 \,. \tag{6.27}$$

6.3 UNFOLDED FREE EQUATIONS: ZERO-FORM SECTOR

In this section, we will consider the unfolded zero-form equations which are the four-dimensional generalizations of (3.57) and read

$$\nabla C = \{\bar e, C\}_\star \,, \tag{6.28}$$

where $\bar{e} := \bar{e}^{\alpha\dot{\alpha}} P_{\alpha\dot{\alpha}}$ and $C(Y|x)$ is a spacetime zero-form given by

$$C(Y|x) = \sum_{\substack{n,\bar{n}=0 \\ n+\bar{n}\in 2\mathbb{N}}}^{\infty} \frac{1}{n!\bar{n}!} C_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_{\bar{n}}}(x)\, y^{\alpha_1}\ldots y^{\alpha_n}\, \bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_{\bar{n}}}\,. \tag{6.29}$$

The restriction $n+\bar{n}\in 2\mathbb{N}$ ensures that $C(Y|x)$ only contains bosonic components. Using the identity

$$\{\bar{e}, \bullet\}_\star = -\frac{i}{l}\, \bar{e}^{\alpha\dot{\alpha}}\, (y_\alpha \bar{y}_{\dot{\alpha}} - \partial_\alpha \partial_{\dot{\alpha}})\, \bullet\,, \tag{6.30}$$

one obtains (in analogy to the derivation of (3.60) in three dimensions)

$$\nabla C_{\alpha(n)\dot{\alpha}(\bar{n})} = -\frac{i}{l}\left(C_{\alpha(n-1)\dot{\alpha}(\bar{n}-1)}\, \bar{e}_{\alpha\dot{\alpha}} - C_{\alpha(n)\beta\dot{\alpha}(\bar{n})\dot{\beta}}\, \bar{e}^{\beta\dot{\beta}}\right)\,, \tag{6.31}$$

where the Lorentz covariant derivative acts on components as

$$\nabla C_{\alpha(n)\dot{\alpha}(\bar{n})} = \mathrm{d}C_{\alpha(n)\dot{\alpha}(\bar{n})} + \bar{\omega}_\alpha{}^\beta C_{\beta\alpha(n-1),\dot{\alpha}(\bar{n})} + \bar{\omega}_{\dot{\alpha}}{}^{\dot{\beta}} C_{\alpha(n)\dot{\beta}\dot{\alpha}(\bar{n}-1)}\,.$$

Equation (6.31) obviously relates only components with the same

$$2s = n - \bar{n}\,. \tag{6.32}$$

We will see shortly that the components $C_{\alpha(2s+n)\dot{\alpha}(n)}$ and $C_{\alpha(n)\dot{\alpha}(2s+n)}$ with $n \geq 0$ and $s \geq 0$ encode spin-s degrees of freedom. Contracting the spacetime index of (6.31) with $\bar{e}^m_{\beta\dot{\beta}}$ and using the identity (6.24) one obtains

$$\nabla_{\beta\dot{\beta}}\, C_{\alpha(n)\dot{\alpha}(\bar{n})} = -\frac{i}{2l}\left(C_{\alpha(n)\beta\dot{\alpha}(\bar{n})\dot{\beta}} - \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\, C_{\alpha(n-1)\dot{\alpha}(\bar{n}-1)}\right)\,, \tag{6.33}$$

Its completely symmetric part with respect to $\beta \leftrightarrow \alpha$ and $\dot{\beta} \leftrightarrow \dot{\alpha}$ gives

$$C_{\alpha(n)\dot{\alpha}(\bar{n})} = 2il\, \frac{1}{n\bar{n}}\, \nabla_{\alpha\dot{\alpha}} C_{\alpha(n-1)\dot{\alpha}(\bar{n}-1)}\,. \tag{6.34}$$

Therefore, the higher components of $C(Y|x)$ are determined in terms of derivatives of the lowest spin-s components

$$C_{\alpha(2s+n)\dot{\alpha}(n)} = \frac{1}{n!n!}\, (2il\, \nabla_{\alpha\dot{\alpha}})^n\, C_{\alpha(2s)}\,, \tag{6.35a}$$

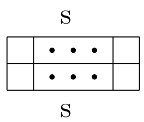
$$C_{\alpha(n)\dot{\alpha}(2s+n)} = \frac{1}{n!n!}\, (2il\, \nabla_{\alpha\dot{\alpha}})^n\, C_{\dot{\alpha}(2s)}\,. \tag{6.35b}$$

For $s = 0$, we can proceed as in three dimensions to see that the lowest component corresponds to a scalar field: we define $\Phi = C(Y = 0)$ and contract (6.33) for $n = \bar{n} = 1$ with $\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$ which results in

$$\nabla^{\beta\dot{\beta}} C_{\beta\dot{\beta}} = \frac{i}{l}\Phi \qquad \Rightarrow \qquad \Box\Phi = -\frac{2}{l^2}\Phi\,. \tag{6.36}$$

We will see that the reality conditions imposed on the zero-form lead to a real scalar field.

For $s > 0$, we can use the dictionary of Section 6.1 to see that $C_{\alpha(2s)}$ and $C_{\dot\alpha(2s)}$ correspond to the Lorentz tensor $C^{a(s),b(s)}$ transforming in a representation of the Lorentz group corresponding to the following Young diagram:



It is therefore natural to assume that they correspond to linearized (higher-spin) Weyl tensors for $s \geq 2$ and the field-strength tensor for $s = 1$. These tensors obey differential Bianchi identities. For example for $s = 1$ these Bianchi identities are given by[1]

$$\nabla_b F^{b,a} = 0\,, \tag{6.37}$$

$$\nabla^a F^{b,c} + \nabla^b F^{c,a} + \nabla^c F^{a,b} = 0\,. \tag{6.38}$$

Let us map these identities into spinorial notation. By (6.10) and (6.8), the equation (6.37) becomes

$$\nabla_{\alpha\dot\alpha}\left(F^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} + F^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\right) = \nabla_\alpha{}^{\dot\beta}F^{\alpha\beta} + \nabla^\beta{}_{\dot\alpha}F^{\dot\beta\dot\alpha} = 0\,. \tag{6.39}$$

Similarly, equation (6.38) is mapped to

$$\nabla^{\gamma\dot\gamma}F^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} + \nabla^{\alpha\dot\alpha}F^{\beta\gamma}\epsilon^{\dot\beta\dot\gamma} + \nabla^{\beta\dot\beta}F^{\gamma\alpha}\epsilon^{\dot\gamma\dot\alpha} + \text{h.c.} = 0\,, \tag{6.40}$$

where h.c. exchanges dotted and undotted indices, e.g. $v^\alpha w^{\dot\alpha} + \text{h.c.} = v^\alpha w^{\dot\alpha} + v^{\dot\alpha} w^\alpha$. Contracting the equation above with $\epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\alpha\gamma}$, one arrives at

$$\nabla^\alpha{}_{\dot\beta}F^{\dot\beta\dot\gamma} - \nabla_\gamma{}^{\dot\gamma}F^{\gamma\alpha} = 0\,. \tag{6.41}$$

Combining this result with (6.39), we obtain the Bianchi identities in spinorial form

$$\nabla_{\beta\dot\alpha}F^\beta{}_\alpha = 0\,, \qquad \nabla_{\alpha\dot\beta}F^{\dot\beta}{}_{\dot\alpha} = 0\,. \tag{6.42}$$

It can be shown that this result generalizes to arbitrary $s \geq 1$: upon defining $C^{\alpha\beta} = F^{\alpha\beta}$ and $C^{\dot\alpha\dot\beta} = F^{\dot\alpha\dot\beta}$ one obtains

$$\nabla_{\beta\dot\alpha}C^\beta{}_{\alpha(2s-1)} = 0\,, \qquad \nabla_{\alpha\dot\beta}C^{\dot\beta}{}_{\dot\alpha(2s-1)} = 0\,. \tag{6.43}$$

1 The case of spin $s = 1$ is in fact degenerate as it encodes also the Maxwell equation in addition to the differential Bianchi identity. For general $s \geq 0$, the corresponding equations are

$$\nabla^{[u}C^{a(s-1)u,b(s-1)u]} = 0 \qquad \nabla_c C^{a(s),b(s-1)c} = 0\,,$$

where in the first equation we antisymmetrize only with respect to the u indices. For $s > 1$ the first equation is the differential Bianchi identity (for symmetric representation of the spin-s Weyl tensor and after imposing the Fronsdal equation) and the second equation follows by taking the trace of the first. For $s = 1$ the two equations are independent.

Let us now see that these equations can indeed be derived from unfolded zero-form equations (6.33). For this consider its lowest components in the spin-s sector

$$\nabla_{\beta\dot{\beta}} C_{\alpha(2s)} \sim C_{\alpha(2s)\beta\dot{\beta}}\,, \tag{6.44}$$

$$\nabla_{\beta\dot{\beta}} C_{\dot{\alpha}(2s)} \sim C_{\dot{\alpha}(2s)\beta\dot{\beta}}\,. \tag{6.45}$$

By contracting these equations with $\epsilon^{\beta\alpha}$ and $\epsilon^{\dot{\beta}\dot{\alpha}}$ respectively, one obtains (6.43) since for example $C_{\alpha(2s)\beta\dot{\beta}}$ is completely symmetric in all its undotted indices.

Let us summarize these points in the following:

- The Y-independent component of $C(Y|x)$ can be identified with a scalar field Φ and the components $C_{\alpha(n)\dot{\alpha}(n)}$ are derivatives thereof. This scalar field is real and its mass term $m^2 = -\frac{1}{2l^2}$ coincides with the value of the conformally coupled scalar.
- The equations of motion for $C(Y|x)$ also encode the Bianchi identities for the (higher-spin) Weyl tensors corresponding to the components

$$C^{\alpha(2s)} \qquad \text{and} \qquad C^{\dot{\alpha}(2s)}. \tag{6.46}$$

  The components

$$C^{\alpha(2s+n)\dot{\alpha}(n)} \qquad \text{and} \qquad C^{\alpha(n)\dot{\alpha}(2s+n)} \tag{6.47}$$

  encode derivatives of these (higher-spin) Weyl tensors.
- We have so far not shown that the unfolded equations (6.28) are only encoding Bianchi identities and a Klein–Gordon equation and no other constraints. We derive this in Appendix C using a cohomological analysis.

The (higher-spin) Weyl tensors should be expressible in terms of the Fronsdal field. In the next section, we will see how this relation is ensured.

6.4 UNFOLDED FREE EQUATIONS: ONE-FORM SECTOR

We have already seen in Section 6.1 that the free fields $\omega^{a(s-1),b(t)}$ of Section 2.2 split in spinorial notation as

$$\omega^{\alpha(s-1+t)\dot{\alpha}(s-1-t)}\,, \qquad \omega^{\alpha(s-1-t)\dot{\alpha}(s-1+t)}\,. \tag{6.48}$$

As in three dimensions, it is convenient to collect them into a single one-form[2] using the oscillators Y_A:

$$\omega(Y|x) = \sum_{s=1}^{\infty} \sum_{t=1-s}^{s-1} \frac{l^{|t|-1}}{(s-1+t)!(s-1-t)!} \times \omega_{\alpha(s-1+t)\dot{\alpha}(s-1-t)}\, y^{\alpha(s-1+t)} \bar{y}^{\dot{\alpha}(s-1-t)} \,. \tag{6.49}$$

We then make the analogous ansatz for the unfolded free equations of motion as in three dimensions

$$D_\Omega\, \omega(Y|x) = 0\,. \tag{6.50}$$

However, as we will discuss, this equation is too strong: it imposes not only Fronsdal equations but also enforces vanishing of the (higher-spin) Weyl tensors. It will require some modification.

The covariant derivative D_Ω is nilpotent (6.27) and therefore (6.50) is invariant under the following gauge transformations

$$\delta\omega(Y|x) = D_\Omega \xi(Y|x) \tag{6.51}$$

for an arbitrary zero-form $\xi(Y|x)$. Using the definition of the covariant derivative (6.25), one easily checks that in components the left hand side of (6.50) becomes

$$\begin{aligned} R_{\alpha(s-1+t)\dot{\alpha}(s-1-t)} :=& \nabla\omega_{\alpha(s-1+t)\dot{\alpha}(s-1-t)} \\ &- \bar{e}_{\dot{\alpha}\beta} \wedge \omega^{\beta}{}_{\alpha(s-1+t)\dot{\alpha}(s-1-t-1)} \\ &- \frac{1}{l^2}\, \bar{e}_{\alpha\dot{\beta}} \wedge \omega_{\alpha(s-1+t-1)\dot{\alpha}(s-1-t)}{}^{\dot{\beta}} \,. \end{aligned} \tag{6.52}$$

By construction, these curvatures are gauge invariant and one can easily check that they have the correct flat limit. The curvatures

$$R^{\alpha(s-1+t)\dot{\alpha}(s-1-t)} \qquad \text{and} \qquad R^{\alpha(s-1-t)\dot{\alpha}(s-1+t)} \tag{6.53}$$

are therefore the spinorial counterparts of $R^{a(s-1),b(t)}$ defined in (2.49).

However, the unfolded equations (6.50) are not quite equivalent to Fronsdal equations. In the unfolded formalism, most components of the fields are expressible in terms of other components. Similarly, most components of the unfolded equations are consequences of other components thereof. As an example, let us recall that, in the zero-form sector, the components $C_{\alpha(s)\dot{\alpha}(\bar{s})}$ of the field $C(Y)$ can be expressed in terms of derivatives of the scalar field or the Weyl tensors by (6.35). Similarly, only a certain component of the unfolded equation (6.28) for the zero-form gives the Klein–Gordon equation or the Bianchi identities (after expressing all components $C_{\alpha(s)\dot{\alpha}(\bar{s})}$ in terms of the scalar

2 The dependence on the AdS radius l can be fixed as follows: it is clear from (2.45) that $\frac{[\omega^{a(s-1),b(t)}]}{[\omega^{a(s-1),b(t+1)}]} = [\text{length}]$. Furthermore we require the (generalized) vielbein $e_n^{a(s-1)}$ to be dimensionless.

and the Weyl tensors). All other components of the unfolded zero-form equation are (differential) consequences thereof. We refer to the Weyl tensors and scalar field as the *dynamical fields* of the zero-form sector. The Bianchi identities (6.43) and Klein–Gordon equation (6.36) are called the *dynamical equations*.

The dynamical fields and equations encoded by the unfolded equations (6.50) of the one-form $\omega(Y|x)$ are most easily determined using a powerful cohomological analysis discussed in Appendix C. We will summarize the results of this analysis in the following.

Let us first discuss the case of fixed spin $s > 1$. The dynamical fields are then encoded in the following components of the one-form

$$\omega_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{e}^{\beta\dot{\beta}}\phi_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} + \bar{e}_{\alpha\dot{\alpha}}\phi'_{\alpha(s-2)\dot{\alpha}(s-2)} + \dots \tag{6.54}$$

By index counting, it is natural to identify $\phi_{\alpha(s)\dot{\alpha}(s)}$ and $\phi'_{\alpha(s-2)\dot{\alpha}(s-2)}$ with the traceless and trace component of the Fronsdal field $\phi_{n(s)}$. All other components of $\omega(Y)$ can be expressed in terms of them through the unfolded equations.

The dynamical equations of the one-form sector are encoded in the following components[3]

$$\begin{aligned}
R_{\alpha(2s-2)} &= -\frac{1}{2}E^{\beta\beta}\mathcal{C}_{\beta\beta\alpha(2s-2)} + \dots \\
&\;\;\vdots \\
R_{\alpha(s)\dot{\alpha}(s-2)} &= E^{\dot{\beta}\dot{\beta}}F_{\alpha(s)\dot{\alpha}(s-2)\dot{\beta}\dot{\beta}} + E_{\alpha\alpha}F'_{\alpha(s-2)\dot{\alpha}(s-2)} + \dots \\
&\;\;. \\
R_{\alpha(s-2)\dot{\alpha}(s)} &= E^{\beta\beta}F_{\alpha(s-2)\beta\beta\dot{\alpha}(s)} + E_{\dot{\alpha}\dot{\alpha}}F'_{\alpha(s-2)\dot{\alpha}(s-2)} + \dots \\
&\;\;\vdots \\
R_{\dot{\alpha}(2s-2)} &= -\frac{1}{2}E^{\dot{\beta}\dot{\beta}}\mathcal{C}_{\dot{\beta}\dot{\beta}\dot{\alpha}(2s-2)} + \dots
\end{aligned} \tag{6.55}$$

By index counting it is natural to identify $F_{\alpha(s)\dot{\alpha}(s)}$ and $F'_{\alpha(s-2)\dot{\alpha}(s-2)}$ with the traceless and trace component of the Fronsdal tensor. However, we see that imposing the unfolded equations (6.50) leads to the constraints $\mathcal{C}_{\alpha(2s)} = 0$ and $\mathcal{C}_{\dot{\alpha}(2s)} = 0$ in addition to the Fronsdal equations. By index counting we are led to interpret these as setting the (higher-spin) Weyl tensors to zero. As in the case of gravity, not all solutions of the Fronsdal equation have vanishing Weyl tensor. Therefore, the unfolded equations (6.50) are not equivalent to Fronsdal equations.

We can turn this bug into a feature by noticing that we interpreted the $C_{\alpha(2s)}$ and $C_{\dot{\alpha}(2s)}$ components of the zero-form $C(y|x)$ as the spin-s Weyl tensor. They should be expressible in terms of the Fronsdal field in order to have this interpretation. This can be ensured by identifying

3 Here we use the definitions $E^{\alpha\alpha} = \bar{e}^{\alpha}{}_{\dot{\beta}} \wedge \bar{e}^{\alpha\dot{\beta}}$ and $E^{\dot{\alpha}\dot{\alpha}} = \bar{e}^{\dot{\alpha}}{}_{\beta} \wedge \bar{e}^{\dot{\alpha}\beta}$.

$\mathcal{C}_{\alpha(2s)} = C_{\alpha(2s)}$ and $\mathcal{C}_{\dot{\alpha}(2s)} = C_{\dot{\alpha}(2s)}$. Doing so we arrive at the following unfolded equations for the one-form $\omega(Y|x)$

$$R_{\alpha(2s-2)} = -\frac{1}{2} E^{\gamma\gamma}\, C_{\alpha(2s-2)\gamma\gamma} \tag{6.56a}$$

$$R_{\alpha(s-1+t)\dot{\alpha}(s-1-t)} = 0 \qquad \text{for } 1-s < t < s-1 \tag{6.56b}$$

$$R_{\dot{\alpha}(2s-2)} = -\frac{1}{2} E^{\dot{\gamma}\dot{\gamma}}\, C_{\dot{\alpha}(2s-2)\dot{\gamma}\dot{\gamma}} \tag{6.56c}$$

These equations impose Fronsdal equations and express the Weyl tensor components of the zero-form in terms of the Fronsdal field.

The case of spin $s = 1$ is slightly degenerate: the dynamical field is given by

$$\omega_0 = A_n \mathrm{d}x^n \,. \tag{6.57}$$

The dynamical equations are encoded in the components

$$R = \mathrm{d}A = -\frac{1}{2} E^{\beta\beta} \mathcal{C}_{\beta\beta} - \frac{1}{2} E^{\dot{\beta}\dot{\beta}} \mathcal{C}_{\dot{\beta}\dot{\beta}} \tag{6.58}$$

and therefore, by identifying $\mathcal{C}_{\beta\beta} = C_{\beta\beta}$ and $\mathcal{C}_{\dot{\beta}\dot{\beta}} = C_{\dot{\beta}\dot{\beta}}$, the field-strength tensor is determined in terms of the Maxwell field A.

We can combine all spins in a compact form using the one-form $\omega(Y|x)$ defined in (6.49):

$$\boxed{D_\Omega\, \omega(Y|x) = -\frac{1}{2} E^{\gamma\gamma}\, \partial_\gamma \partial_\gamma\, C(y,0|x) - \frac{1}{2} E^{\dot{\gamma}\dot{\gamma}}\, \partial_{\dot{\gamma}} \partial_{\dot{\gamma}}\, C(0,\bar{y}|x)\,.} \tag{6.59}$$

This relation is known as the *central on-mass-shell theorem* [58, 59] and plays a crucial role in the construction of Vasiliev equations.

In this section we have motivated the identification of the Fronsdal operator and the (generalized) Weyl tensors by index counting. In Appendix C, it is explained how this identification can be shown rigorously.

6.5 SUMMARY: FREE UNFOLDED EQUATIONS

As in the three-dimensional case, it is convenient to define the twisted adjoint covariant derivative

$$\begin{aligned} \tilde{D}_\Omega F :=& \mathrm{d}F - \Omega \wedge \star F + (-1)^{|F|}\, F \wedge \star\, \pi(\Omega) \\ =& \nabla F - \frac{1}{l} \bar{e}^{\alpha\dot{\alpha}} \left(y_\alpha \bar{y}_{\dot{\alpha}} - \partial_{\dot{\alpha}} \partial_\alpha \right) F\,, \end{aligned} \tag{6.60}$$

where we defined

$$\pi(y_\alpha, \bar{y}_{\dot{\alpha}}) = (y_\alpha, -\bar{y}_{\dot{\alpha}})\,. \tag{6.61}$$

Therefore $\pi(P_{\alpha\dot{\alpha}}) = -P_{\alpha\dot{\alpha}}$ while $L_{\alpha\alpha}$ and $L_{\dot{\alpha}\dot{\alpha}}$ are left invariant. This allows us to rewrite the free unfolded equations in a compact form

$$\tilde{D}_\Omega\, C(Y|x) = 0\,, \tag{6.62a}$$

$$D_\Omega\, \omega(Y|x) = -\frac{1}{2} E^{\gamma\gamma}\, \partial_\gamma \partial_\gamma\, C(y,0|x) - \frac{1}{2} E^{\dot{\gamma}\dot{\gamma}}\, \partial_{\dot{\gamma}} \partial_{\dot{\gamma}}\, C(0,\bar{y}|x)\,. \tag{6.62b}$$

These equations are invariant under the following gauge transformations

$$\boxed{\begin{aligned}\delta C(Y|x) &= 0\,, \\ \delta\omega(Y|x) &= D_\Omega \xi(Y|x)\,.\end{aligned}} \qquad \begin{aligned}(6.63\text{a})\\(6.63\text{b})\end{aligned}$$

The zero-form $C(Y|x)$ contains a real scalar field and the (higher-spin) Weyl tensors. Furthermore $C(Y|x)$ also contains their derivatives. The one-form $\omega(Y|x)$ contains the spinorial analogs of the Fronsdal fields. Its unfolded equations (6.62b) impose Fronsdal equations and express the (higher-spin) Weyl tensors in terms of the Fronsdal fields. This is achieved by the source term in (6.62b) which links the zero-form and one-form fields.

6.6 VASILIEV EQUATIONS

In this section, we will introduce four-dimensional Vasiliev equations which reduce to the free unfolded equations of the last section upon linearizing around an AdS_4-background. *For notational simplicity we will work in units for which $l = 1$.*

As we have already seen for the three-dimensional case, Vasiliev theory is formulated in terms of masterfields $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{S}_A$. The masterfields $\mathcal{W}$ and $\mathcal{B}$ contain the fields $\omega(Y|x)$ and $C(Y|x)$ respectively in addition to auxiliary fields. As in three dimensions, these auxiliary fields arise from introducing a set of oscillators Z_A in addition to Y_A. The star product for functions depending on both sets of oscillators is then given by

$$\begin{aligned}(f\star g)(Y,Z) = \tfrac{1}{(2\pi)^4}\int \mathrm{d}^4U\,\mathrm{d}^4V\; f(Y+U, Z+U)&\\ \times g(Y+V, Z-V)\exp\left(iV^A U_A\right).&\end{aligned} \tag{6.64}$$

The masterfields are of the following form

$$\mathcal{W}(Y,Z|x) = \omega(Y|x) + f(Y,Z|x)\,, \tag{6.65}$$

$$\mathcal{B}(Y,Z|x) = C(Y|x) + g(Y,Z|x)\,, \tag{6.66}$$

$$\mathcal{S}_A(Y,Z|x) = f_A(Y,Z|x)\,, \tag{6.67}$$

where the functions f, g and f_A vanish for $Z_A = 0$. *Vasiliev equations* in four dimensions then read

$$\boxed{\begin{array}{ll}\mathrm{d}\mathcal{W} = \mathcal{W}\wedge\star\mathcal{W}\,, & \\ \mathrm{d}(\mathcal{B}\star\varkappa) = [\mathcal{W}, \mathcal{B}\star\varkappa]_\star\,, & \\ \mathrm{d}\mathcal{S}_\alpha = [\mathcal{W}, \mathcal{S}_\alpha]_\star\,, & \mathrm{d}\bar{\mathcal{S}}_{\dot\alpha} = [\mathcal{W}, \bar{\mathcal{S}}_{\dot\alpha}]_\star\,, \\ [\mathcal{S}_\alpha, \mathcal{S}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + e^{i\theta}\mathcal{B}\star\varkappa)\,, & \\ [\mathcal{S}_{\dot\alpha}, \mathcal{S}_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1 + e^{-i\theta}\mathcal{B}\star\bar\varkappa)\,, & \\ [\mathcal{S}_\alpha, \mathcal{S}_{\dot\alpha}]_\star = 0\,. & \\ \{\mathcal{S}_\alpha, \mathcal{B}\star\varkappa\}_\star = 0\,, & \{\mathcal{S}_{\dot\alpha}, \mathcal{B}\star\bar\varkappa\}_\star = 0\,,\end{array}} \qquad \begin{array}{r}(6.68\text{a})\\(6.68\text{b})\\(6.68\text{c})\\(6.68\text{d})\\(6.68\text{e})\\(6.68\text{f})\\(6.68\text{g})\end{array}$$

and are invariant under the following gauge transformations

$$\boxed{\begin{aligned} \delta\mathcal{W} &= \mathrm{d}\xi - [\mathcal{W},\xi]_\star\,, \\ \delta(\mathcal{B}\star\varkappa) &= [\xi,\mathcal{B}\star\varkappa]_\star\,, \\ \delta\mathcal{S}_\alpha &= [\xi,\mathcal{S}_\alpha]_\star\,, \qquad \delta\mathcal{S}_{\dot\alpha} = [\xi,\mathcal{S}_{\dot\alpha}]_\star\,. \end{aligned}} \quad \begin{aligned}(6.69a)\\(6.69b)\\(6.69c)\end{aligned}$$

In the relations above, we have defined the Kleinians

$$\varkappa := e^{iz_\alpha y^\alpha}\,, \qquad \bar\varkappa := e^{i\bar z_{\dot\alpha}\bar y^{\dot\alpha}}\,, \tag{6.70}$$

which obey

$$\varkappa\star F(Y,Z)\star\varkappa = F(\pi(Y),\pi(Z))\,, \tag{6.71a}$$

$$\bar\varkappa\star F(Y,Z)\star\bar\varkappa = F(\bar\pi(Y),\bar\pi(Z))\,, \tag{6.71b}$$

where we have defined

$$\pi(y_\alpha,\bar y_{\dot\alpha},z_\alpha,\bar z_{\dot\alpha}) := (-y_\alpha,\bar y_{\dot\alpha},-z_\alpha,\bar z_{\dot\alpha})\,, \tag{6.72a}$$

$$\bar\pi(y_\alpha,\bar y_{\dot\alpha},z_\alpha,\bar z_{\dot\alpha}) := (y_\alpha,-\bar y_{\dot\alpha},z_\alpha,-\bar z_{\dot\alpha})\,. \tag{6.72b}$$

The *bosonic theory* is defined by imposing the following constraints on the masterfields

$$[\mathcal{W},K]_\star = 0\,, \qquad [\mathcal{B},K]_\star = 0\,, \qquad \{\mathcal{S}_A,K\}_\star = 0\,, \tag{6.73}$$

where we have defined the *total Kleinian* by $K := \varkappa\star\bar\varkappa$. Note that from $\varkappa\star\bar\varkappa\star\varkappa = \bar\varkappa$ and $\varkappa\star\varkappa = 1$, it follows that $K = \bar\varkappa\star\varkappa$. We will exclusively consider the bosonic theory in the following.

The masterfields obey the following reality conditions [60]

$$\mathcal{W}^\dagger = -\mathcal{W}\,, \qquad (S_\alpha)^\dagger = -S_{\dot\alpha}\,, \qquad \mathcal{B}^\dagger = \mathcal{B}\,. \tag{6.74}$$

where hermitian conjugation of the oscillators is defined by

$$(y_\alpha)^\dagger = \bar y_{\dot\alpha}\,, \qquad (z_\alpha)^\dagger = -\bar z_{\dot\alpha}\,. \tag{6.75}$$

Similar to the three-dimensional case, this ensures that the (generalized) spin-connections and vielbeins are real and that $\mathcal{B}$ contains a real scalar field.

The θ parameter is related to possible invariance of the Vasiliev equations under parity. It can be shown that parity maps (6.68d) to (see for example Section 2.1.11 of [4]):

$$[\mathcal{S}_{\dot\alpha},\mathcal{S}_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1+e^{i\theta}P(\mathcal{B})\star\bar\varkappa)\,. \tag{6.76}$$

Similarly, (6.68e) transforms as

$$[\mathcal{S}_\alpha,\mathcal{S}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1+e^{-i\theta}P(\mathcal{B})\star\varkappa)\,, \tag{6.77}$$

where $P(\mathcal{B})$ denotes the parity transformed field $\mathcal{B}$. This masterfield can either be parity even or odd [4]. Vasiliev equations can therefore only be parity invariant for

$$P(\mathcal{B}) = \mathcal{B} \qquad \rightarrow \qquad \theta = 0\,,$$
$$P(\mathcal{B}) = -\mathcal{B} \qquad \rightarrow \qquad \theta = \tfrac{\pi}{2}\,.$$

It can be shown that these choices indeed lead to parity invariance of all other Vasiliev equations. The first possibility is called *type A Vasiliev theory*. It contains a parity even scalar field encoded in the masterfield $\mathcal{B}$. The second case is *type B Vasiliev theory* which contains a pseudo-scalar. Let us also mention in passing that there is also a so called *minimal truncation* for these models in which only gauge fields with even spin are contained in the theory [3]. For other values of the phase θ the interactions of the theory violate the parity symmetry.

Minimal Type A was conjectured in [8] to be dual to the free or critical three-dimensional $O(N)$ vector model depending on the boundary conditions for the scalar field. Similarly, it was conjectured in [9] that minimal type B is dual to N free fermions with $O(N)$ symmetry in 3d or the 3d Gross-Nevau model depending on the choice of boundary conditions for the scalar.

6.7 LORENTZ COVARIANT PERTURBATION THEORY

The Vasiliev equations of the last section can be solved exactly by

$$\mathcal{W}^{(0)} = \Omega\,, \qquad \mathcal{B}^{(0)} = 0\,, \qquad \mathcal{S}_A^{(0)} = Z_A\,, \tag{6.78}$$

where Ω is the AdS$_4$ connection defined in (6.22). This statement can be easily checked by using analogous arguments as for the three-dimensional case discussed in Section 3.2.2. Vasiliev equations then allow us to perturbatively extract equations of motion for the physical fields propagating on an AdS$_4$ background. However, the set of equations (6.68) will not lead to manifestly local Lorentz covariant equations in complete analogy to the three-dimensional case. Following the same arguments as outlined for the three-dimensional case in Section 5.2, one can ensure local Lorentz covariance with respect to the background fields by a field redefinition of the background spin-connection

$$\bar{\omega}^{\alpha\alpha} L^Y_{\alpha\alpha} \rightarrow \bar{\omega}^{\alpha\alpha}\left(L^Y_{\alpha\alpha} + L^Z_{\alpha\alpha} - L^S_{\alpha\alpha}\right) = \bar{\omega}^{\alpha\alpha}\hat{L}_{\alpha\alpha}\,, \tag{6.79}$$
$$\bar{\omega}^{\dot{\alpha}\dot{\alpha}} L^Y_{\dot{\alpha}\dot{\alpha}} \rightarrow \bar{\omega}^{\dot{\alpha}\dot{\alpha}}\left(L^Y_{\dot{\alpha}\dot{\alpha}} + L^Z_{\dot{\alpha}\dot{\alpha}} - L^S_{\dot{\alpha}\dot{\alpha}}\right) = \bar{\omega}^{\dot{\alpha}\dot{\alpha}}\hat{L}_{\dot{\alpha}\dot{\alpha}}\,, \tag{6.80}$$

where we have used the definitions

$$L^Y_{AB} = -\frac{i}{4}\{Y_A, Y_B\}_\star\,,$$
$$L^Z_{AB} = \frac{i}{4}\{Z_A, Z_B\}_\star\,,$$
$$L^S_{AB} = \frac{i}{4}\{\mathcal{S}_A, \mathcal{S}_B\}_\star\,,$$

together with

$$\hat{L}_{AB} = L^Y_{AB} + L^Z_{AB} - L^S_{AB}\,.$$

We will therefore expand around the following background one-form

$$\hat{\Omega} = \frac{1}{2}\bar{\omega}^{\alpha\alpha}\hat{L}_{\alpha\alpha} + \frac{1}{2}\bar{\omega}^{\dot{\alpha}\dot{\alpha}}\hat{L}_{\dot{\alpha}\dot{\alpha}} + \bar{e}^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}}\,. \tag{6.81}$$

As in three dimensions, it is advantageous to shift all fields by their vacuum values

$$\mathcal{S}_\alpha \to Z_A + 2i\mathcal{A}_A\,, \qquad \mathcal{W} \to \hat{\Omega} + W\,, \qquad \mathcal{B} \to 2i\,\mathcal{B}\,. \tag{6.82}$$

Imposing the Schwinger–Fock gauge

$$Z^A\mathcal{A}_A = 0\,, \tag{6.83}$$

one then obtains from the non-dynamical[4] Vasiliev equations (6.68c)-(6.68g) after some straightforward algebra

$$\partial^Z_A\mathcal{W} = -[\bar{e}, \mathcal{A}_A]_\star + \chi_A\,, \tag{6.84a}$$
$$\partial^Z_A\mathcal{B} = \mathcal{A}_A \star \mathcal{B} + \mathcal{B}\star\bar{\mathcal{A}}_A\,, \tag{6.84b}$$
$$\partial^Z_A\mathcal{A}_B - \partial^Z_B\mathcal{A}_A = [\mathcal{A}_A, \mathcal{A}_B]_\star + R_{AB} \tag{6.84c}$$

where we have defined

$$\bar{\mathcal{A}}_A = \begin{pmatrix} \varkappa\star\mathcal{A}_\alpha\star\varkappa \\ \bar{\varkappa}\star\mathcal{A}_{\dot{\alpha}}\star\bar{\varkappa} \end{pmatrix}, \tag{6.85a}$$

$$R_{AB} = \begin{pmatrix} \epsilon_{\alpha\beta}\, e^{+i\theta}\,\mathcal{B}\star\varkappa & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}}\, e^{-i\theta}\,\mathcal{B}\star\bar{\varkappa} \end{pmatrix}, \tag{6.85b}$$

and χ_A is a function which obeys $Z^A\chi_A = 0$ and whose explicit form will be of no importance as will become clear momentarily. This is useful as the differential equations[5]

$$\partial^Z_A f(Y,Z) = g_A(Y,Z)\,,$$
$$\partial^Z_A f_B(Y,Z) - \partial^Z_B f_A(Y,Z) = g_{AB}(Y,Z)\,,$$

are respectively solved by

$$f(Y,Z) = \epsilon(Y) + Z^A\Gamma_0\langle g_A\rangle\,, \tag{6.86a}$$
$$f_A(Y,Z) = \partial^Z_A\xi(Y,Z) - Z^B\Gamma_1\langle g_{AB}\rangle\,, \tag{6.86b}$$

where the homotopy integrals are defined by

$$\Gamma_n\langle f\rangle(Z) := \int_0^1 \mathrm{d}t\, t^n\, f(tZ)\,. \tag{6.87}$$

4 By non-dynamical Vasiliev equations we refer to those Vasiliev equations which determine the Z-dependence of the masterfields. This terminology was introduced in Section 3.2 and is unrelated to the dynamical equations in the sense of the σ_--cohomology.

5 The functions g_A and g_{AB} have to obey the compatibility conditions $\partial^Z_A g_B - \partial^Z_B g_A \equiv 0$ and $\partial_C g_{AB} + \partial_A g_{BC} + \partial_B g_{CA} = 0$ respectively. However, for expressions considered here, this will automatically hold due to the consistency of Vasiliev equations.

Therefore, the contribution proportional to χ_A will drop out upon solving (6.84a) which explains why we did not spell out its explicit form. We can use these results to formally determine the Z-dependence of all masterfields in Schwinger–Fock gauge

$$\begin{aligned}
\mathcal{W} &= \omega(Y) - z^{\alpha}\Gamma_0\langle[\bar{e}+\mathcal{W},\mathcal{A}_{\alpha}]_{\star}\rangle + \text{h.c.}\,,\\
\mathcal{B} &= C(Y) + z^{\alpha}\Gamma_0\langle\mathcal{A}_{\alpha}\star\mathcal{B}+\mathcal{B}\star\pi(\mathcal{A}_{\alpha})\rangle + \text{h.c.}\,,\\
\mathcal{A}_{\alpha} &= z_{\alpha}\Gamma_1\langle\mathcal{A}_{\delta}\star\mathcal{A}^{\delta}\rangle + \bar{z}^{\dot{\beta}}\Gamma_1\langle[\mathcal{A}_{\dot{\beta}},\mathcal{A}_{\alpha}]_{\star}\rangle + z_{\alpha}\Gamma_1\langle\mathcal{B}\star\varkappa\rangle e^{+i\theta}\,,\\
\mathcal{A}_{\dot{\alpha}} &= \bar{z}_{\dot{\alpha}}\Gamma_1\langle\mathcal{A}_{\dot{\delta}}\star\mathcal{A}^{\dot{\delta}}\rangle + z^{\beta}\Gamma_1\langle[\mathcal{A}_{\beta},\mathcal{A}_{\dot{\alpha}}]_{\star}\rangle + \bar{z}_{\dot{\alpha}}\Gamma_1\langle\mathcal{B}\star\bar{\varkappa}\rangle e^{-i\theta}\,.
\end{aligned} \tag{6.88}$$

The h.c.-operation exchanges barred with unbarred variables and flips the sign of θ but does not conjugate complex numbers. We will use this notation repeatedly in the following.

Similarly, one can derive the shifted dynamical equations from (6.68a) and (6.68b) which, after some algebra and using the equations of motion of the background vielbein and spin-connection, reads

$$D^{yz}\mathcal{W} = \mathcal{W}\star\wedge\mathcal{W} - \frac{1}{2}\,E^{\alpha\alpha}\,L^{S}_{\alpha\alpha} - \frac{1}{2}\,E^{\dot{\alpha}\dot{\alpha}}\,L^{S}_{\dot{\alpha}\dot{\alpha}} + \chi \tag{6.89a}$$

$$\tilde{D}^{yz}\mathcal{B} = W\star\mathcal{B} - \mathcal{B}\star\pi(\mathcal{W})\,, \tag{6.89b}$$

where χ vanishes for $Z=0$ and will therefore not contribute to the equations for the physical fields. Furthermore, we have introduced the covariant derivatives

$$D^{yz}\bullet := \nabla^{yz} - \bar{e}^{\alpha\dot{\alpha}}[P_{\alpha\dot{\alpha}},\bullet]_{\star}\,, \tag{6.90a}$$

$$\tilde{D}^{yz}\bullet := \nabla^{yz} - \bar{e}^{\alpha\dot{\alpha}}\{P_{\alpha\dot{\alpha}},\bullet\}_{\star}\,, \tag{6.90b}$$

where we have also defined

$$\begin{aligned}
\nabla^{yz}\bullet &= \mathrm{d}\bullet - \frac{1}{2}\bar{\omega}^{\alpha\alpha}[L^{0}_{\alpha\alpha},\bullet]_{\star} - \frac{1}{2}\bar{\omega}^{\dot{\alpha}\dot{\alpha}}[L^{0}_{\dot{\alpha}\dot{\alpha}},\bullet]_{\star}\\
&= \mathrm{d}\bullet + \bar{\omega}^{\alpha\alpha}(y_{\alpha}\partial^{y}_{\alpha} + z_{\alpha}\partial^{z}_{\alpha})\bullet + \bar{\omega}^{\dot{\alpha}\dot{\alpha}}(\bar{y}_{\dot{\alpha}}\partial^{y}_{\dot{\alpha}} + \bar{z}_{\dot{\alpha}}\partial^{z}_{\dot{\alpha}})\bullet\,.
\end{aligned} \tag{6.91}$$

with $L^{0}_{AB} = L^{Y}_{AB} + L^{Z}_{AB}$. Using the definition of the star product, one can easily show that $\bar{e} = \bar{e}^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}}$ acting on a spacetime zero-form is

$$[\bar{e},\bullet]_{\star} = \bar{e}^{\alpha\dot{\alpha}}\,\{(y_{\alpha} - i\partial^{z}_{\alpha})\partial^{y}_{\dot{\alpha}} + (\bar{y}_{\dot{\alpha}} - i\partial^{z}_{\dot{\alpha}})\partial^{y}_{\alpha})\}\bullet\,, \tag{6.92a}$$

$$\{\bar{e},\bullet\}_{\star} = -i\,\bar{e}^{\alpha\dot{\alpha}}\,\{(y_{\alpha} - i\partial^{z}_{\alpha})(\bar{y}_{\dot{\alpha}} - i\partial^{z}_{\dot{\alpha}}) - \partial^{y}_{\alpha}\partial^{y}_{\dot{\alpha}}\}\bullet\,. \tag{6.92b}$$

Following analogous arguments as for the three-dimensional case, one can then show that Lorentz covariance with respect to the background fields will be manifest. One could again straightforwardly extend the manifest Lorentz covariance beyond the background fields but we will not do so for the same reasons as in three dimensions.

6.8 LINEAR ORDER

Expanding (6.88) to linear order gives

$$\mathcal{W}^{(1)} = \omega^{(1)}(Y) + M\,, \tag{6.93a}$$

$$\mathcal{B}^{(1)} = C^{(1)}(Y)\,, \tag{6.93b}$$

$$\mathcal{A}^{(1)}_{\beta} = z_\beta \Gamma_1 \langle C^{(1)} \star \varkappa \rangle e^{i\theta} = z_\beta \int_0^1 \mathrm{d}t\, t\, C^{(1)}(-tz, \bar{y}) e^{ityz+i\theta}\,, \tag{6.93c}$$

$$\mathcal{A}^{(1)}_{\dot\beta} = \bar{z}_\beta \Gamma_1 \langle C^{(1)} \star \bar\varkappa \rangle e^{-i\theta} = \bar{z}_\beta \int_0^1 \mathrm{d}t\, t\, C^{(1)}(y, -t\bar{z}) e^{it\bar{y}\bar{z}-i\theta}\,, \tag{6.93d}$$

where we defined

$$M = -z^\alpha \Gamma_0 \langle [\bar{e}, \mathcal{A}^{(1)}_\alpha]_\star \rangle + \text{h.c.}\,. \tag{6.94}$$

Using that for $n \neq m$ the identity $\Gamma_n \circ \Gamma_m = -(\Gamma_n - \Gamma_m)/(n-m)$ holds, this can be simplified and one obtains

$$M = -i\, \bar{e}^{\alpha\dot\alpha} z_\alpha \partial^y_{\dot\alpha} \int_0^1 \mathrm{d}t\,(1-t)\, C^{(1)}(-zt, \bar{y}) e^{ityz+i\theta} + \text{h.c.}\,, \tag{6.95}$$

which we will then insert in the dynamical first order Vasiliev equations obtained by expanding (6.89) to linear order

$$D^{yz}\mathcal{W}^{(1)}\Big|_{Z=0} = 0\,, \qquad \tilde{D}^{yz}\mathcal{B}^{(1)}\Big|_{Z=0} = 0\,. \tag{6.96}$$

This results in the following equations of motion

$$D\omega^{(1)} = \mathcal{V}(\Omega, \Omega, C)\,, \qquad \tilde{D}C^{(1)} = 0\,, \tag{6.97}$$

where we defined

$$\mathcal{V}(\Omega,\Omega,C) = [\bar{e}, M]\Big|_{Z=0} = -\frac{1}{2} E^{\dot\alpha\dot\alpha} \partial^y_{\dot\alpha} \partial^y_{\dot\alpha} C^{(1)}(0,\bar{y}) e^{i\theta} + \text{h.c.}\,. \tag{6.98}$$

This precisely coincides with the free equations (6.62) for $\theta = 0$.[6] Therefore, Vasiliev theory provides us with a non-linear "completion" of these free equations.

6 We merely restricted to $\theta = 0$ in our discussion of the free equations for simplicity. The on-mass-shell theorem defines the generalized Weyl tensors in terms of the Fronsdal fields. Setting $\theta \neq 0$ only changes this definition by a phase.

6.9 SECOND ORDER

Expanding (6.88) to second order and using the explicit form (6.93a) of $\mathcal{W}^{(1)}$ gives

$$\mathcal{B}^{(2)} = C^{(2)}(Y) + B'^{(2)}\,, \tag{6.99a}$$

$$\mathcal{A}^{(2)}_{\alpha} = z_{\alpha}\Gamma_1\langle\mathcal{A}^{(1)}_{\gamma} \star \mathcal{A}^{(1)\gamma}\rangle + \bar{z}^{\dot{\beta}}\Gamma_1\langle[\mathcal{A}^{(1)}_{\dot{\beta}}, \mathcal{A}^{(1)}_{\alpha}]_{\star}\rangle$$
$$+ z_{\alpha}\Gamma_1\langle\mathcal{B}^{(2)} \star \varkappa\rangle e^{i\theta}\,, \tag{6.99b}$$

$$\mathcal{A}^{(2)}_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}}\Gamma_1\langle\mathcal{A}^{(1)}_{\dot{\delta}} \star \mathcal{A}^{(1)\dot{\delta}}\rangle + z^{\beta}\Gamma_1\langle[\mathcal{A}^{(1)}_{\beta}, \mathcal{A}^{(1)}_{\dot{\alpha}}]_{\star}\rangle$$
$$+ \bar{z}_{\dot{\alpha}}\Gamma_1\langle\mathcal{B}^{(2)} \star \bar{\varkappa}\rangle e^{-i\theta}\,, \tag{6.99c}$$

$$\mathcal{W}^{(2)} = \omega^{(2)}(Y) + M^{(2)} - z^{\alpha}\Gamma_0\langle[\omega^{(1)}, \mathcal{A}^{(1)}_{\alpha}]\rangle$$
$$- z^{\alpha}\Gamma_0\langle[M, \mathcal{A}^{(1)}_{\alpha}]\rangle + \text{h.c.}\,, \tag{6.99d}$$

where we have defined

$$M^{(2)} = - z^{\alpha}\Gamma_0\langle[\bar{e}, \mathcal{A}^{(2)}_{\alpha}]_{\star}\rangle - \text{h.c.}\,, \tag{6.100a}$$

$$B'^{(2)} = z^{\alpha}\Gamma_0\langle\mathcal{A}^{(1)}_{\alpha} \star C^{(1)} + C^{(1)} \star \pi(\mathcal{A}^{(1)}_{\alpha})\rangle + \text{h.c.}\,. \tag{6.100b}$$

The second order dynamical equations then read

$$D^{yz}\mathcal{W}^{(2)} = (\omega^{(1)} + M) \star\wedge (\omega^{(1)} + M) - i\,E^{\alpha\alpha}\,\mathcal{A}^{(1)}_{\alpha} \star\wedge \mathcal{A}^{(1)}_{\alpha}$$
$$- i\,E^{\dot{\alpha}\dot{\alpha}}\mathcal{A}^{(1)}_{\dot{\alpha}} \star\wedge \mathcal{A}^{(1)}_{\dot{\alpha}}\,, \tag{6.101a}$$

$$\tilde{D}^{yz}\mathcal{B}^{(2)} = \omega^{(1)} \star C^{(1)} - C^{(1)} \star \pi(\omega^{(1)}) + M \star C^{(1)}$$
$$- C^{(1)} \star \pi(M)\,, \tag{6.101b}$$

where we implicitly set $Z = 0$ after having performed all the star products. Using analogous techniques as in the three-dimensional case, one then obtains the second order equations of motion for the physical fields

$$D\omega^{(2)} - \mathcal{V}(\Omega, \Omega, C^{(2)}) = \mathcal{V}(\omega, \omega) + \mathcal{V}(\Omega, \omega, C)$$
$$+ \mathcal{V}(\Omega, \Omega, C, C)\,, \tag{6.102a}$$

$$\tilde{D}C^{(2)} = \mathcal{V}(\omega, C) + \mathcal{V}(\Omega, C, C)\,, \tag{6.102b}$$

These source terms are most conveniently given in Fourier space. We will use the wave-vectors $\xi^A = (\xi^{\alpha}, \bar{\xi}^{\dot{\alpha}})$ and $\eta^A = (\eta^{\alpha}, \bar{\eta}^{\dot{\alpha}})$ along with the following convention for Fourier transformed fields

$$f(Y) = \int \mathrm{d}^4\xi\, f(\xi) e^{iY^A\xi_A}\,. \tag{6.103}$$

The source terms purely given in terms of star products then read

$$\begin{aligned}\mathcal{V}(\omega,\omega) &= \omega^{(1)} \wedge \star\, \omega^{(1)} \\ &= \int \mathrm{d}^4\xi\, \mathrm{d}^4\eta\, e^{i((y-\eta)(y+\xi)+\text{h.c.}}\, \omega^{(1)}(\xi) \wedge \omega^{(1)}(\eta)\,, \\ \mathcal{V}(\omega,C) &= \omega^{(1)} \star C^{(1)} - C^{(1)} \star \pi(\omega^{(1)}) \\ &= \int \mathrm{d}^4\xi\, \mathrm{d}^4\eta \Big(e^{i(y-\eta)(y+\xi)+\text{h.c.}}\, \omega^{(1)}(\xi)\, C^{(1)}(\eta) \\ &\qquad - e^{i(y+\eta)(y+\xi)+\text{h.c.}}\, C^{(1)}(\xi)\, \omega^{(1)}(\eta) \Big)\,,\end{aligned}$$

while all other source terms take the form[7]

$$\begin{aligned}\mathcal{V}(\Omega,\Omega,C^{(2)}) &= \int d^2\xi \left(E^{\alpha\alpha} T_{\alpha\alpha} + \text{h.c.} \right) C^{(2)}(\xi)\,, \\ \mathcal{V}(\Omega,C,C) &= \int d^2\xi\, d^2\eta\, (\bar{e}^{\alpha\alpha} K_{\alpha\alpha} + \text{h.c.})\, C^{(1)}(\xi) C^{(1)}(\eta)\,, \\ \mathcal{V}(\Omega,\omega,C) &= \int d^2\xi\, d^2\eta \Big(\bar{e}^{\alpha\dot{\alpha}} L_{\alpha\dot{\alpha}}\, \omega^{(1)}(\xi) C^{(1)}(\eta) \\ &\qquad + \bar{e}^{\alpha\dot{\alpha}} \bar{L}_{\dot{\alpha}\dot{\alpha}}\, C^{(1)}(\xi)\, \omega^{(1)}(\eta) \Big)\,, \\ \mathcal{V}(\Omega,\Omega,C,C) &= \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta \left(E^{\alpha\alpha} J_{\alpha\alpha} + E^{\dot{\alpha}\dot{\alpha}} J_{\dot{\alpha}\dot{\alpha}} + \text{h.c.} \right) \\ &\qquad \times C^{(1)}(\xi) C^{(1)}(\eta)\,.\end{aligned}$$

The kernels are then given by

$$\boxed{\begin{aligned} T_{\alpha\alpha} &= \frac{1}{2}\xi_\alpha \xi_\alpha e^{i(y\xi-\theta)}\,, \\ K_{\alpha\dot{\alpha}} &= \int_0^1 \mathrm{d}t \Big((\bar{y}_{\dot{\alpha}} t - (1-t)\bar{\xi}_{\dot{\alpha}})\eta_\alpha\, R_2 - (\bar{y}_{\dot{\alpha}} t + (1-t)\bar{\eta}_{\dot{\alpha}})\xi_\alpha\, S_1 \Big) \\ L_{\alpha\dot{\alpha}} &= \int_0^1 \mathrm{d}t \left(R_1\, \xi_\alpha (\bar{\eta}_{\dot{\alpha}} + t\bar{\xi}_{\dot{\alpha}}) + \text{h.c.} \right), \\ \bar{L}_{\alpha\dot{\alpha}} &= \int_0^1 \mathrm{d}t \left(R_2\, \eta_\alpha (\bar{\xi}_{\dot{\alpha}} + t\bar{\eta}_{\dot{\alpha}}) + \text{h.c.} \right), \\ J_{\alpha\alpha} &= \int_0^1 \mathrm{d}t\, \mathrm{d}q \left((y+\xi)_\alpha (y+\eta)_\alpha \left(iq^2t^2 + (\bar{\xi}\bar{\eta})\tfrac{qt(1-qt)}{2} \right) Q_1 \right), \\ J_{\dot{\alpha}\dot{\alpha}} &= \int_0^1 \mathrm{d}t\, \mathrm{d}q \Big(-\frac{i}{2}\bar{\xi}_{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}}\, Q_1 + \frac{i}{2}(1-t)\bar{\xi}_{\dot{\alpha}}\eta_{\dot{\alpha}}\, P_1 + \frac{i}{2}\partial^{\bar{y}}_{\dot{\alpha}}\partial^{\bar{y}}_{\dot{\alpha}}\, K_0 \Big)\,, \end{aligned}}$$

while the phases R_1, R_2, S_1, Q_1, P_1 and K_0 read

$$\begin{aligned} R_1 &= \exp i\left((y(1-t) - t\eta)\xi + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + \theta \right), \\ R_2 &= \exp i\left((y(1-t) - t\xi)\eta + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + \theta \right), \\ S_1 &= \exp i\left((y(1-t) + t\eta)\xi + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + \theta \right), \\ Q_1 &= \exp i\left((qt(y+\eta)(y+\xi) + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + 2\theta \right), \\ P_1 &= \exp i\left((t(y+\eta)(y+\xi) + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + 2\theta \right), \\ K_0 &= \exp i\left(t\eta\xi + (\bar{y} - \bar{\eta})(\bar{y} + \bar{\xi}) + 2\theta \right). \end{aligned}$$

7 In these expressions, the Fourier transformation is only with respect to y (or $\bar{y}$) and we suppress the dependence on $\bar{y}$ (or y), for example $C^{(1)}(\eta) = \int \mathrm{d}^2y\, C^{(1)}(y,\bar{y})\, e^{iy\eta}$.

These expressions are significantly more compact than their three dimensional counterparts. Similarly to three dimensional case, nilpotency of D and $\tilde{D}$ imposes consistency conditions on the source terms. For example acting with $\tilde{D}$ on (6.102b) gives

$$\tilde{D}\,\{\mathcal{V}(\omega, C) + \mathcal{V}(\Omega, C, C)\} = 0\,. \tag{6.104}$$

We checked that all these consistency conditions are fulfilled for our results (see Appendix B.4 of our publication [27] for details).

6.10 EXTRACTING CORRECTIONS TO FRONSDAL EQUATION

In this section, we will extract the corrections to the Fronsdal equations induced by the scalar field.

To this end, it is useful to introduce the following notation: consider the number operators $N = y^\nu \partial_\nu$ and $\bar{N} = \bar{y}^{\dot\nu}\partial_{\dot\nu}$. For a function $f(Y)$ we denote its component with $N - \bar{N} = 2k$ by

$$f_k(Y) = \sum_{s=k+1}^{\infty} \tfrac{1}{(s-1+k)!(s-1-k)!} f_{\alpha(s-1+k)\dot\alpha(s-1-k)}\, y^{\alpha(s-1+k)}\, \bar{y}^{\dot\alpha(s-1-k)}\,.$$

Furthermore, we will decompose the covariant derivative D as follows

$$D = \nabla + Q\,, \tag{6.105}$$

where Q is given by

$$Q = y^\alpha \bar{e}_\alpha{}^{\dot\alpha}\partial_{\dot\alpha} + \bar{y}^{\dot\alpha}\bar{e}^\alpha{}_{\dot\alpha}\partial_\alpha =: Q_+ + Q_-\,. \tag{6.106}$$

From our discussion in Section 6.4, it is clear that the corrections to the Fronsdal equations can be extracted from the second order one-form equation of motion (6.102a) by considering the following components

$$R_0^{(2)\prime} := \nabla e^{(2)} + Q_+\omega_{-1}^{(2)} + Q_-\omega_{+1}^{(2)} = J_0\,, \tag{6.107a}$$

$$R_{+1}^{(2)\prime} := \nabla\omega_{+1}^{(2)} + Q_+ e^{(2)} + Q_-\omega_{+2}^{(2)} = J_{+1}\,, \tag{6.107b}$$

$$R_{-1}^{(2)\prime} := \nabla\omega_{-1}^{(2)} + Q_+\omega_{-2}^{(2)} + Q_- e^{(2)} = J_{-1}\,, \tag{6.107c}$$

where we have denoted $e^{(2)} = \omega_0^{(2)}$ and $J = \mathcal{V}(\Omega, \Omega, C, C)$. We have dropped all contributions from the (generalized) Weyl tensors since they do not contribute to the Fronsdal equation as was also discussed in Section 6.4. Furthermore, we neglect all contribution from the term $\mathcal{V}(\omega, \omega)$ because we are interested in corrections induced by the scalar field only.

Similarly to our discussion of the three-dimensional case in Section 5.6.3, we need to solve the torsion constraint (6.107a) for $\omega_{\pm 1}^{(2)}$. This can be done as follows: the one-form $\omega^{(2)}$ can uniquely be decomposed as

$$\begin{aligned}\omega^{(2)} = \;& \bar{e}^{\alpha\dot\alpha}\,\partial_\alpha\partial_{\dot\alpha}\,\omega^{\partial\partial} + Q_+\,\omega^{Q_+} \\ & + \bar{e}^{\alpha\dot\alpha}\,y_\alpha\bar{y}_{\dot\alpha}\,\omega^{y\bar{y}} + Q_-\,\omega^{Q_-}\,.\end{aligned} \tag{6.108}$$

Similarly, the two-form J has the following decomposition

$$\begin{aligned} J = \;& E^{\alpha\alpha}\partial_\alpha\partial_\alpha J^{\partial\partial} + E^{\alpha\alpha}y_\alpha\partial_\alpha J^{y\partial} + E^{\alpha\alpha}y_\alpha y_\alpha J^{yy} \\ & + E^{\dot\alpha\dot\alpha}\partial_{\dot\alpha}\partial_{\dot\alpha}\bar J^{\partial\partial} + E^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\partial_{\dot\alpha}\bar J^{y\partial} + E^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\bar y_{\dot\alpha}\bar J^{yy}\,. \end{aligned} \tag{6.109}$$

One can easily check that $Q_+\omega^{(2)}$ and $Q_-\omega^{(2)}$ are given by

$$\begin{aligned} Q_+\omega^{(2)} &= \frac{N}{2}E^{\dot\alpha\dot\alpha}\partial_{\dot\alpha}\partial_{\dot\alpha}\omega^{\partial\partial} - \frac{\bar N+2}{2}E^{\alpha\alpha}y_\alpha y_\alpha\,\omega^{y\bar y} \\ &\quad + \frac{1}{2}\left[-NE^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\partial_{\dot\alpha} + (\bar N+2)E^{\alpha\alpha}y_\alpha\partial_\alpha\right]\omega^{Q_-}\,, \\ Q_-\omega^{(2)} &= \frac{\bar N}{2}E^{\alpha\alpha}\partial_\alpha\partial_\alpha\omega^{\partial\partial} - \frac{N+2}{2}E^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\bar y_{\dot\alpha}\,\omega^{y\bar y} \\ &\quad + \frac{1}{2}\left[-\bar NE^{\alpha\alpha}y_\alpha\partial_\alpha + (N+2)E^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\partial_{\dot\alpha}\right]\omega^{Q_+}\,. \end{aligned} \tag{6.110}$$

From this one can straightforwardly invert the relation

$$Q_+\omega^{(2)}_{-1} + Q_-\omega^{(2)}_{+1} = J_0 \tag{6.111}$$

by choosing[8]

$$\begin{aligned} \omega^{\partial\partial}_{+1} &= \frac{2}{\bar N}J_0^{\partial\partial}\,, & \omega^{y\bar y}_{+1} &= \frac{-2}{(N+2)}\bar J_0^{yy}\,, \\ \omega^{Q_+}_{+1} &= \frac{1}{N+\bar N+2}(NJ_0^{y\partial} + (\bar N+2)\bar J_0^{y\partial})\,, \end{aligned} \tag{6.112}$$

together with

$$\begin{aligned} \omega^{\partial\partial}_{-1} &= \frac{2}{N}\bar J_0^{\partial\partial}\,, & \omega^{y\bar y}_{-1} &= \frac{-2}{(\bar N+2)}J_0^{yy}\,, \\ \omega^{Q_-}_{-1} &= \frac{1}{N+\bar N+2}(\bar N\bar J_0^{y\partial} + (N+2)J_0^{y\partial})\,. \end{aligned} \tag{6.113}$$

We will denote these as

$$\omega^{(2)}_{+1} := Q_-^{\#}J_0\,, \qquad \omega^{(2)}_{-1} := Q_+^{\#}J_0\,. \tag{6.114}$$

We note that the expressions above only degenerate for the spin-1 case for which there is no torsion constraint to be solved for.

These results can be used to solve the generalized torsion constraint (6.107a) by defining

$$\omega^{(2)}_{-1} = \omega^{(2)}_{-1}(e) + Q_+^{\#}J_0\,, \tag{6.115}$$

$$\omega^{(2)}_{+1} = \omega^{(2)}_{+1}(e) + Q_-^{\#}J_0\,, \tag{6.116}$$

8 For $k \in \mathbb{N}$ the inverse number operators are given by the homotopy integrals

$$(N+k)^{-1}f(y) = \int_0^1 \mathrm{d}t\, t^{k-1}\, f(ty)\,.$$

where $\omega^{(2)}_{\pm 1}(e)$ denote the solutions for vanishing torsion as the system (6.107) is now given by

$$R_0^{(2)} := \nabla e^{(2)} + Q_+\omega^{(2)}_{-1}(e) + Q_-\omega^{(2)}_{+1}(e) = 0\,, \tag{6.117a}$$
$$R_{+1}^{(2)} := \nabla \omega^{(2)}_{+1}(e) + Q_+ e^{(2)} + Q_-\omega^{(2)}_{+2} = j_{+1}\,, \tag{6.117b}$$
$$R_{-1}^{(2)} := \nabla \omega^{(2)}_{-1}(e) + Q_+ \omega^{(2)}_{-2} + Q_- e^{(2)} = j_{-1}\,, \tag{6.117c}$$

with the Fronsdal current defined as

$$j_{\pm 1} := J_{\pm 1} - \nabla Q^{\#}_{\mp} J_0\,. \tag{6.118}$$

We are now in a position to straightforwardly extract the corrections to the Fronsdal equation by performing the following projections

$$R^{(2)}_{+1}\Big|_F := E^{\dot\alpha\dot\alpha}\partial_{\dot\alpha}\partial_{\dot\alpha}F + E^{\alpha\alpha}y_\alpha y_\alpha F'\,,$$
$$R^{(2)}_{-1}\Big|_F := E^{\alpha\alpha}\partial_{\alpha}\partial_{\alpha}F + E^{\dot\alpha\dot\alpha}\bar y_{\dot\alpha}\bar y_{\dot\alpha} F'\,,$$

where F and F' denote the traceless part and trace of the Fronsdal operators respectively.[9] From this we see explicitly that $R^{(2)}_{+1}|_F$ and $R^{(2)}_{-1}|_F$ carry the same information about the Fronsdal tensor. In the following we will therefore restrict to $R^{(2)}_{+1}|_F$ sector which leads to the equation

$$R^{(2)}_{+1}|_F = j_{+1}|_F\,. \tag{6.121}$$

The only difficulty left is to decompose the Fronsdal current (6.118) as follows

$$j_{+1}|_F = E^{\dot\alpha\dot\alpha}\partial_{\dot\alpha}\partial_{\dot\alpha}\,\mathsf{j} + E^{\alpha\alpha}y_\alpha y_\alpha\,\mathsf{j}'\,, \tag{6.122}$$

where j and j' correspond to the source terms of the traceless and trace component of the corrected Fronsdal equation respectively. This can be easily done by using the following identity

$$E^{\alpha\alpha}g_{\alpha\alpha}(y) = E^{\alpha\alpha}\Big(\partial_\alpha\partial_\alpha \tfrac{1}{N(N-1)}y^\beta y^\beta g_{\beta\beta} + y_\alpha\partial_\alpha \tfrac{2}{N(N+2)}y^\beta\partial_\gamma g_\beta{}^\gamma + y_\alpha y_\alpha \tfrac{1}{(N+2)(N+3)}\partial_\beta\partial_\beta g^{\beta\beta}\Big)\,. \tag{6.123}$$

9 This can be seen as follows: we expand F and F' as follows

$$F = \sum_s \frac{1}{s!s!}\,F_{\alpha(s),\dot\alpha(s)}\,y^{\alpha(s)}\bar y^{\dot\alpha(s)}\,,$$
$$F' = \sum_s \frac{1}{(s-2)!(s-2)!}\,F'_{\alpha(s-2),\dot\alpha(s-2)}\,y^{\alpha(s-2)}\bar y^{\dot\alpha(s-2)}\,.$$

By comparing with (6.55), we see that the components $F_{\alpha(s),\dot\alpha(s)}$ and $F'_{\alpha(s-2),\dot\alpha(s-2)}$ encode the traceless and trace component of the Fronsdal tensor.

Applying this identity to (6.118), we obtain the components of Fronsdal current in terms of the source term J

$$\begin{aligned}
\mathsf{j} =& -\frac{1}{\bar N(\bar N-1)}\partial_\nu \nabla^{\nu\dot\nu}\bar y_{\dot\nu} J^{\partial\partial} + \bar J^{\partial\partial} \\
&- \frac{1}{2\bar N(\bar N+N)} y_\nu \bar y_{\dot\nu}\nabla^{\nu\dot\nu} J^{\diamond}\,, && (6.124)\\
\mathsf{j}' =& -\frac{1}{(N+2)(N+3)}\partial_\nu \nabla^{\nu\dot\nu}\bar y_{\dot\nu}\bar J^{yy} + J^{yy} \\
&+ \frac{1}{2(N+2)(\bar N+N+4)}\partial_\nu\partial_{\dot\nu}\nabla^{\nu\dot\nu} J^{\diamond}\,, && (6.125)
\end{aligned}$$

where we defined $J^\diamond := NJ^{y\partial} + (\bar N+2)\bar J^{y\partial}$. Since we calculated the source term J in Section 6.9, we can now easily extract the corresponding Fronsdal current. This allows us to analyze the corrections induced by the scalar field.

6.11 EXPECTATION FOR THE RESULT

As was discussed in Section 5.6.5, it can be shown that the part of Fronsdal current $\mathsf{j}^{\min}_{n(s)}(\Phi^\dagger,\Phi)$ in (5.76), which is bilinear in the scalar field $\Phi := C(Y=0)$, can be brought into a form which contains only up to s number of derivatives. This minimal current corresponds to[10]

$$\begin{aligned}
j^s(y,\bar y) &= C^{(1)}(y,\bar y)C^{(1)}(-y,\bar y)\Big|_{\text{spin s and scalar}} \\
&= \sum_{k=0}^{s} \tfrac{1}{s!s!}\tfrac{(-1)^k (s!)^2}{(k!)^2((s-k)!)^2}\, C_{\alpha(s-k)\dot\alpha(s-k)}\, C_{\alpha(k)\dot\alpha(k)}\, y^{\alpha(s)}\,\bar y^{\dot\alpha(s)}
\end{aligned} \tag{6.126}$$

in the language of four dimensional Vasiliev theory. This statement can be verified along similar lines as in Section 5.6.5 as we will discuss in the following. One immediately concludes that (6.126) only contains up to s derivatives of the scalar since $C_{\alpha(n)\dot\alpha(n)} \sim (\nabla_{\alpha\dot\alpha})^n\Phi$. In Fourier space (6.126) is given by

$$\begin{aligned}
&\int \mathrm{d}^4\xi \mathrm{d}^4\eta\, e^{iy(\xi-\eta)+i\bar y(\bar\xi+\bar\eta)}\, C^{(1)}(\xi)C^{(1)}(\eta)\Big|_{\text{spin s and scalar}} \\
&= \int \mathrm{d}^4\xi \mathrm{d}^4\eta \sum_{k=0}^{s} \frac{(-1)^k}{(k!)^2((s-k)!)^2}\,(y\xi\,\bar y\bar\xi)^{s-k}(y\eta\,\bar y\bar\eta)^k\, C^{(1)}(\xi)C^{(1)}(\eta)\,.
\end{aligned} \tag{6.127}$$

Using the equations of motion for $C^{(1)}$ in Fourier space, one can check that this expression is conserved

$$\nabla^{\alpha\dot\alpha}\partial_\alpha\partial_{\dot\alpha}\, j^s(y,\bar y) = 0\,. \tag{6.128}$$

10 By $\bullet\big|_{\text{spin s and scalar}}$ we denote the projection on terms only involving $C^{(1)}_{\alpha(n)\dot\alpha(\bar n)}$ with $n=\bar n$ and s number of y and $\bar y$ oscillators.

For showing this, the relative sign in the y_α-oscillators is essential. As in three dimensions, we therefore conclude that $j^s(y,\bar{y})$ corresponds to

$$j^s(y,\bar{y}) = \frac{1}{s!s!} \mathrm{j}^{\min}_{\alpha(s)\dot{\alpha}(s)}\, y^{\alpha(s)} \bar{y}^{\dot{\alpha}(s)}\,, \tag{6.129}$$

where $\mathrm{j}^{\min}_{\alpha(s)\dot{\alpha}(s)}$ is the spinorial equivalent of the minimal current $j^{\min}_{n(s)}$. We therefore expect the following second order relations

$$\begin{aligned} R^{(2)}_{\alpha(s)\dot{\alpha}(s-2)}\Big|_F &= E^{\dot{\beta}\dot{\beta}} F_{\alpha(s)\dot{\alpha}(s-2)\dot{\beta}\dot{\beta}} + E_{\alpha\alpha} F'_{\alpha(s-2)\dot{\alpha}(s-2)} \\ &= a_s\, E^{\dot{\beta}\dot{\beta}} \mathrm{j}^{\min}_{\alpha(s)\dot{\alpha}(s-2)\dot{\beta}\dot{\beta}}\,. \end{aligned} \tag{6.130}$$

We would like to extract the coefficients a_s from Vasiliev theory. However, as we will discuss in the next section, this will again confront us with the problem of pseudo-local field redefinitions similar to our discussion of the three-dimensional case.

6.12 EXPLICIT RESULTS

In this section, we will summarize some explicit results for the Fronsdal current. The spin-2 sector is given by

$$R^{(2)}_{\alpha\alpha}\Big|_F = 2\cos(2\theta) \left(E^{\dot{\alpha}\dot{\alpha}}\, \mathrm{j}_{\alpha\alpha\dot{\alpha}\dot{\alpha}} + E_{\alpha\alpha}\, \mathrm{j}' \right)\,. \tag{6.131}$$

where we have defined

$$\begin{aligned} \mathrm{j}_{\alpha(2)\dot{\alpha}(2)} = \sum_{l=0}^{\infty} \Big(& a_{l,1}\, C_{\alpha\nu(l)\dot{\alpha}\dot{\nu}(l)}\, C_{\alpha}{}^{\nu(l)}{}_{\dot{\alpha}}{}^{\dot{\nu}(l)} \\ & + 2a_{l,0}\, C_{\alpha(2)\nu(l)\dot{\alpha}(2)\dot{\nu}(l)}\, C^{\nu(l)\dot{\nu}(l)} \Big)\,, \end{aligned} \tag{6.132}$$

$$\mathrm{j}' = \sum_{l=0}^{\infty} c_{l,0}\, C_{\nu(l)\dot{\nu}(l)}\, C^{\nu(l)\dot{\nu}(l)}\,. \tag{6.133}$$

The coefficients take the following form

$$a_{l,0} = \frac{1}{l!l!}\left(-\frac{3}{(2+l)^2} + \frac{7}{2(2+l)} - \frac{4}{3+l} + \frac{1}{2(4+l)} \right)\,, \tag{6.134a}$$

$$a_{l,1} = \frac{1}{l!l!}\left(\frac{1}{2(2+l)^2} - \frac{1}{4(2+l)} + \frac{1}{4(4+l)} \right)\,, \tag{6.134b}$$

$$c_{l,0} = \frac{1}{l!l!}\left(\frac{1}{12(1+l)^2} - \frac{3}{8(1+l)} + \frac{1}{2+l} - \frac{1}{8(3+l)} \right)\,. \tag{6.134c}$$

As can be seen from these results, the Fronsdal current is again of pseudo-local form as was the case for the three-dimensional case discussed in Section 5.6.6.

The coefficients for arbitrary spin are rather involved. Similarly to the three-dimensional case, one can again show that the backreaction of the scalar field splits in various independently conserved sectors [27].

One can restrict to the sector containing the minimal current. The resulting expressions are slightly more compact

$$R^{(2)}_{\alpha(s)\dot\alpha(s-2)}\Big|_F = 2\cos(2\theta)\, E^{\dot\beta\dot\beta}\, \mathrm{j}_{\alpha(s)\alpha(s-2)\dot\beta\dot\beta}\,, \tag{6.135}$$

with the current components taking the following form

$$\mathrm{j}_{\alpha(s)\dot\alpha(s)} = \sum_{l=0}^{\infty}\sum_{k=0}^{s} a^{(s)}_{l,k}\, C_{\alpha(s-k)\nu(l)\dot\alpha(s-k)\dot\nu(l)}\, C_{\alpha(k)}{}^{\nu(l)}{}_{\dot\alpha(k)}{}^{\dot\nu(l)}\,, \tag{6.136}$$

with the coefficients

$$a^{(s)}_{l,k} = \frac{(-)^k s!s!}{l!l!k!k!(s-k)!(s-k)!}\,\frac{s(2l(s-1)+s(2s-1))}{8(s-1)(l+s)^2(l+s+1)^2}\,. \tag{6.137}$$

Note that there is no trace contribution as we have restricted to the minimal sector.

It is important to note that from an analysis of conformal field theories with (slightly-broken) higher-spin symmetry [61], one would expect $\cos^2(\theta)$ dependence which obviously differs from the factor $\cos(2\theta)$ of our results. We will return to this puzzling observation in the discussion section.

7 PSEUDO-LOCAL FIELD REDEFINITIONS

As was mentioned in Section 6.11, the source term involving two scalar fields is expected to be of the form[1]

$$\mathrm{j}^{\min}_{\alpha(s)\dot\alpha(s)} = a_s \sum_{k=0}^{s} c_{s,k}\, C_{\alpha(s-k)\dot\alpha(s-k)}\, C_{\alpha(k)\dot\alpha(k)}\,, \tag{7.1}$$

where we have used the definition of the minimal current (6.126) and defined

$$c_{s,k} = \frac{(-1)^k s!s!}{k!k!(s-k)!(s-k)!}\,. \tag{7.2}$$

The coefficient a_s denotes the spin-dependent normalization constant which we want to extract from Vasiliev theory.

In the minimal current sector our explicit result (6.136) obtained from Vasiliev theory can be rearranged as follows

$$\mathrm{j}_{\alpha(s)\dot\alpha(s)} = \sum_{l=0}^{\infty} \frac{1}{l!l!} a_{l,s} \sum_{k=0}^{s} c_{s,k}\, C_{\alpha(s-k)\nu(l)\dot\alpha(s-k)\dot\nu(l)}\, C_{\alpha(k)}{}^{\nu(l)}{}_{\dot\alpha(k)}{}^{\dot\nu(l)}\,, \tag{7.3}$$

where we defined

$$a_{l,s} = \frac{s(2l(s-1)+s(2s-1))}{8(s-1)(l+s)^2(l+s+1)^2}\,. \tag{7.4}$$

To rewrite these expressions in a more suggestive way let us define

$$\nabla_{m(s-k)n(l)}\Phi\, \nabla_{m(k)}{}^{n(l)}\Phi := \bar e_m^{\alpha\dot\alpha} \ldots \bar e_m^{\alpha\dot\alpha}\, C_{\alpha(s-k)\nu(l)\dot\alpha(s-k)\dot\nu(l)}\, C_{\alpha(k)}{}^{\nu(l)}{}_{\dot\alpha(k)}{}^{\dot\nu(l)}\,. \tag{7.5}$$

In accordance with our discussion in Section 6.1 the left hand side of the above equation denotes the traceless and completely symmetric combinations of $s-k+l$ and $k+l$ derivatives acting on the first and second scalar field respectively of which l pairs of derivatives are contracted. We will call the tensor structure

$$\sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)}\Phi\, \nabla_{m(k)}\Phi \tag{7.6}$$

1 For the following discussion it is more convenient to include the spin-dependent normalization a_s in the definition of the minimal current.

the *minimal interaction term*. Note that this is precisely the tensor structure of the minimal current $\mathrm{j}^{\min}_{n(s)}$. All other structures involving contractions of derivatives

$$\sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)n(l)}\Phi\, \nabla_{m(k)}{}^{n(l)}\Phi \qquad \text{with } l>0 \tag{7.7}$$

will be referred to as *successors* of the minimal interaction term.

In Minkowski space, this corresponds to the fact that on top of the minimal conserved tensor $\Phi \overleftrightarrow{\partial}_n \dots \overleftrightarrow{\partial}_n \Phi$ there is a family of conserved successors $\partial_{m(l)}\Phi \overleftrightarrow{\partial}_n \dots \overleftrightarrow{\partial}_n \partial^{m(l)}\Phi$ that have $2l$ derivatives contracted.[2] Note also that successors have to be distinguished from *improvement terms* which we will only use for terms which are conserved without using the equations of motion.

In order to extract the spin-dependent constant a_s in $\mathrm{j}^{\min}_{\alpha(s)\dot\alpha(s)}$, we therefore need to remove all successors from the result (7.3) obtained from Vasiliev theory by field redefinitions and then read off the coefficient in front of its minimal interaction term afterwards. In order to gain some intuition about these field redefinitions, we will however first consider a toy model in the following section.

7.1 A PSEUDOLOCAL TOY MODEL

We will study a toy model of two scalar fields Ψ and Φ in Minkowski space. First let us consider the action

$$\begin{aligned} S = \int \mathrm{d}^D x \Big(& \Phi(\Box - m^2)\Phi - \Psi(\Box - M^2)\Psi - a_0 \Phi^2 \Psi \\ & - a_1 (\partial_n \Phi \partial^n \Phi)\Psi + \dots \Big)\,, \end{aligned} \tag{7.8}$$

where we have only written out terms up to cubic order in the scalar fields. The equations of motion are then given by

$$(\Box - M^2)\Psi = a_0 \Phi^2 + a_1 (\partial_n \Phi \partial^n \Phi) + \dots\,, \tag{7.9}$$

$$(\Box - m^2)\Phi = 2a_0 \Phi\Psi + 2a_1 \partial_n \Phi \partial^n \Psi + 2a_1 \Psi \Box \Phi + \dots\,. \tag{7.10}$$

Using the field redefinition

$$\Psi \to \Psi + \frac{1}{2} a_1 \Phi^2 \tag{7.11}$$

one can remove the term proportional to a_1 in the equation of motion for Ψ. This can be seen by considering the redefined field equation for Ψ

$$(\Box - M^2)\Psi = a_0 \Phi^2 + a_1 (\partial_n \Phi \partial^n \Phi) - \frac{1}{2} a_1 (\Box - M^2)\Phi^2 + \dots\,. \tag{7.12}$$

2 Here we use the notation $\overleftrightarrow{\partial}_n := \overleftarrow{\partial}_n - \overrightarrow{\partial}_n$.

The last term can be simplified by using

$$\begin{aligned}\Box\Phi^2 &= 2(\partial_n\Phi\partial^n\Phi) + 2\Phi\Box\Phi \\ &= 2(\partial_n\Phi\partial^n\Phi) + 2m^2\Phi^2 + \dots\,,\end{aligned} \tag{7.13}$$

where we have used that we can neglect all interaction terms in the equation of motion of Φ since we are only interested in terms involving up to two scalar fields in the relation above. Therefore, equation (7.12) indeed leads to

$$(\Box - M^2)\Psi = \left(a_0 + \left(\tfrac{1}{2}M^2 - m^2\right) a_1\right) \Phi^2 + \dots\,. \tag{7.14}$$

In the language introduced in the last section the Φ^2 term constitutes the minimal interaction term while the $\partial_n\Phi\partial^n\Phi$ is a successor thereof. We can remove the successor by a field redefinition - at the expense of modifying the prefactor of the minimal term.

Let us now extend this toy model a bit further and consider the following equations of motion

$$(\Box - M^2)\Psi = a_0\Phi^2 + \sum_{l=0}^{L+1} a_l\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi + \dots\,, \tag{7.15}$$

$$(\Box - m^2)\Phi = \dots\,, \tag{7.16}$$

where we have not written out explicitly terms involving three scalar fields in the first equation of motion and terms involving *two* scalar fields in the second equation of motion. Furthermore, we have denoted l partial derivatives ∂_n by $\partial_{n(l)}$ to ease notation.

We will now consider a field redefinition of the form

$$\Psi \to \Psi + \sum_{l=0}^{L} b_l\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi\,. \tag{7.17}$$

We want to fix the coefficients b_l in such a way that only the minimal interaction term proportional to Φ^2 is left in (7.15) after the redefinition - possibly with a corrected prefactor. One can derive the following identity

$$\Box(\partial_{n(l)}\Phi\, \partial^{n(l)}\Phi) = 2m^2\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi + 2\partial_{n(l+1)}\Phi\, \partial^{n(l+1)}\Phi\,, \tag{7.18}$$

where we have used that to the relevant order the equation of motion of Φ is given by $\Box\Phi = m^2\Phi$. Using this identity we obtain

$$\begin{aligned}(\Box - M^2) &\sum_{l=0}^{L} b_l\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi \\ &= b_0(2m^2 - M^2)\Phi^2 + 2b_L\, \partial_{n(L+1)}\Phi\, \partial^{n(L+1)}\Phi \\ &\quad + \sum_{l=1}^{L} \left(b_l(2m^2 - M^2) + 2b_{l-1}\right)\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi\,.\end{aligned} \tag{7.19}$$

In order to cancel all successors in (7.15), we first study how to remove a single successor with $L+1$ contractions and unit coefficient. To this end let us consider (7.15) with

$$a_{L+1} = 1\,, \qquad a_L = 0\,, \qquad a_{L-1} = 0\,, \qquad \ldots \qquad a_1 = 0\,. \tag{7.20}$$

By (7.19) we then obtain the following conditions for the coefficients b_l

$$\begin{aligned} a_{L+1} &= 1 = 2b_L\,, \\ a_L &= 0 = b_L(2m^2 - M^2) + 2b_{L-1}\,, \\ &\vdots \\ a_1 &= 0 = b_1(2m^2 - M^2) + 2b_0\,. \end{aligned}$$

This recursive relation has the solution

$$b_l = \tfrac{1}{2}(\tfrac{1}{2}M^2 - m^2)^{L-l} \qquad \text{with } 0 \leq l \leq L\,. \tag{7.21}$$

By (7.19) this leads to the following shift in the prefactor of the minimal term

$$a_0 \to a_0 - b_0(2m^2 - M^2) = a_0 + C_{L+1}\,, \tag{7.22}$$

where we have defined

$$C_l := (\tfrac{1}{2}M^2 - m^2)^l\,. \tag{7.23}$$

This result allows us to determine the contribution to the minimal term due to the removal of the successor

$$a_l\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi \tag{7.24}$$

by replacing $C_{L+1} \to C_l a_l$ in (7.22). Consecutively removing each successor in (7.15) then leads to the following equation of motion for Ψ

$$(\Box - M^2)\Psi = \tilde{a}_0 \Phi^2\,, \tag{7.25}$$

where we have neglected cubic terms and defined

$$\tilde{a}_0 := \sum_{l=0}^{L+1} C_l a_l\,. \tag{7.26}$$

Turning this discussion around, we therefore see that we can encode a certain coefficient $\tilde{a}_0$ in front of the minimal interaction term by adding an arbitrary number of successors.

Coming from Vasiliev theory, it is natural to go one step further: instead of encoding the coefficient of the minimal term by an arbitrary but finite number of successors we will now encode it using infinitely many successors. Therefore, we consider the following equations of motion

$$(\Box - M^2)\Psi = a_0\Phi^2 + \sum_{l=0}^{\infty} a_l\, \partial_{n(l)}\Phi\, \partial^{n(l)}\Phi + \ldots\,, \tag{7.27}$$

$$(\Box - m^2)\Phi = \ldots\,. \tag{7.28}$$

From our previous construction, we deduce by considering the partial sums of the interaction term above that there exists a field redefinition

$$\Psi \to \Psi + \sum_{l=0}^{\infty} b_l \, \partial_{n(l)} \Phi \, \partial^{n(l)} \Phi \, . \tag{7.29}$$

with suitable choices for the coefficients b_l such that after performing this field redefinition the equations of motion for Ψ are given by

$$(\Box - M^2)\Psi = \tilde{a}_0 \Phi^2 \, , \tag{7.30}$$

where we have again neglected cubic terms and defined

$$\tilde{a}_0 := \sum_{l=0}^{\infty} C_l a_l \, . \tag{7.31}$$

If this sum converges, the equation of motion (7.27) is merely an equivalent (albeit complicated) way of rewriting (7.30). Notice in particular that the redefinition (7.29) contains an infinite number of derivatives and is therefore of pseudo-local form. However, this does not constitute a problem as it is merely a manifestation of the fact that we chose to encode the coefficient $\tilde{a}_0$ of the minimal interaction term by an infinite number of successors. Indeed, it is clear from our discussion that one necessarily needs to perform pseudo-local field redefinitions of the type constructed above in order to project on the minimal interaction term.

The pseudo-local field redefinition considered above does not allow us to remove all interaction terms. The minimal interaction term cannot further be removed using the algorithm outlined above. This is because these field redefinitions necessarily lead to terms involving at least two derivatives - as can be seen by (7.13). However, one can easily construct a different class of pseudo-local field redefinitions which allows us to remove also this minimal term. To this end, consider

$$\psi \to \psi + \tilde{a}_0 (\Box - M^2)^{-1} \Phi^2 \, . \tag{7.32}$$

The last term in this redefinition can formally be rewritten as

$$\tilde{a}_0 (\Box - M^2)^{-1} \Phi^2 = -\tilde{a}_0 \, M^2 \sum_{n=0}^{\infty} \left(\frac{\Box}{M^2} \right)^n \Phi^2 \, . \tag{7.33}$$

By induction, one can straightforwardly show that

$$\Box^n \Phi^2 = \sum_{k=0}^{n} (2m^2)^{n-k} \, 2^k \binom{n}{k} \partial_{m(k)} \Phi \, \partial^{m(k)} \Phi \, , \tag{7.34}$$

and therefore the redefinition is given by

$$\Psi \to \Psi - \sum_{n=0}^{\infty} \sum_{k=0}^{n} M^{-2n+2} \, (2m^2)^{n-k} \, 2^k \binom{n}{k} \partial_{m(k)} \Phi \, \partial^{m(k)} \Phi \, . \tag{7.35}$$

Obviously, this type of pseudo-local field redefinition cannot be physically permitted as they allow us to remove all interactions[3] and therefore change the observables of the field theory.

Let us summarize the lessons learned from studying this toy model:

- Not all pseudo-local field redefinitions are physically permitted because a subset of them allows for a complete removal of the interactions.
- On the other hand, if some coefficient in front of the minimal term is encoded by an infinite number of successors certain pseudo-local field redefinitions are necessarily required in order to project on the minimal interaction term.
- We gave an explicit algorithm which allows us to project on the minimal interaction term and read off its coefficient by constructing field redefinitions which iteratively remove any number (possibly infinite) of successors.

In the next section, we will apply these lessons to Vasiliev theory. This will only require a minimal generalization of our arguments due to the fact that the spin-s source terms contain s free spacetime indices and that Vasiliev theory is formulated on an AdS-background instead of Minkowski space.

7.2 ANALYSIS FOR VASILIEV THEORY

In the following, we will discuss how the Fronsdal current (7.3), which in vectorial notation reads

$$\mathrm{j}_{m(s)} = \sum_{l=0}^{\infty} \frac{1}{l!l!}\, a_{l,s} \sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)n(l)}\Phi\, {\nabla_{m(k)}}^{n(l)}\Phi\,, \tag{7.36}$$

can be projected on the minimal interaction term (7.1)

$$\mathrm{j}^{\min}_{m(s)} = \tilde{a}_0^{(s)} \sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)}\Phi\, \nabla_{m(k)}\Phi\,, \tag{7.37}$$

which we have also rewritten in vectorial notation. Performing this projection would allow us to extract the coefficient $\tilde{a}_0^{(s)}$ from Vasiliev theory.

In complete analogy to the discussion in the last section, we will first study how one can remove a single successor with $(L+1)$ contractions and unit coefficient, i.e.

$$\sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)n(L+1)}\Phi\, {\nabla_{m(k)}}^{n(L+1)}\Phi\,, \tag{7.38}$$

---

3 Strictly speaking the constructed pseudo-local field redefinition only allows to remove all interactions bilinear in the scalar field Φ but it is clear that one can easily generalize the construction to the case of arbitrary interactions since it involves inverting the kinetic operator $\Box - M^2$.

by a field redefinition

$$\phi_{m(s)} \to \phi_{m(s)} + \delta\phi_{m(s)}\,, \tag{7.39}$$

where we have defined

$$\delta\phi_{m(s)} = \sum_{l=0}^{L} \frac{1}{l!l!}\, b_l^{(s)} \sum_{k=0}^{s} c_{s,k}\, \nabla_{m(s-k)n(l)}\Phi\, \nabla_{m(k)}{}^{n(l)}\Phi\,. \tag{7.40}$$

From our discussion in Section 6.4 it is clear, that the Fronsdal field is embedded into $\omega_0^{(2)}(Y)$ as follows

$$\omega^{(2)}_{\alpha(s-1),\dot\alpha(s-1)} = \bar e^{\beta\dot\beta}\phi_{\beta\alpha(s-1)\dot\beta\dot\alpha(s-1)} + \bar e_{\alpha\alpha}\phi'_{\alpha(s-2)\dot\alpha(s-2)} + \dots\,, \tag{7.41}$$

where $\phi_{\alpha(s)\dot\alpha(s)}$ and $\phi'_{\alpha(s-2)\dot\alpha(s-2)}$ denote the traceless and trace components of the Fronsdal field respectively and we have not made any other components explicit. It is important to emphasize that, since we are working at the second order, it is not clear if this component indeed corresponds to the second order Fronsdal field. We will return to this point in the next section where we also show that this assumption can be relaxed.

Since the redefinition (7.40) is traceless by construction, we see that

$$\omega^{(2)}_{\alpha(s)\dot\alpha(s)} \to \omega^{(2)}_{\alpha(s)\dot\alpha(s)} + \bar e^{\beta\dot\beta}\delta\phi_{\beta\alpha(s-1)\dot\beta\dot\alpha(s-1)}\,, \tag{7.42}$$

where $\delta\phi_{\alpha(s)\dot\alpha(s)}$ denotes the spinorial analog of (7.40). From our discussion of the minimal interaction term in Section 6.11, it is clear that using Y_A oscillators this can be rewritten as

$$\omega_0^{(2)}(Y) \to \omega_0^{(2)}(Y) + \bar e^{\alpha\dot\alpha}\partial_\alpha\partial_{\dot\alpha}\,\delta\omega_0^{(2)}(Y)\,, \tag{7.43}$$

with

$$\delta\omega_0^{(2)} \sim \int \mathrm{d}^4\xi\,\mathrm{d}^4\eta \sum_{l=0}^{\infty} b_l^{(s)} \sum_{k=0}^{s} c_{s,k}\,(y\xi\,\bar y\bar\xi)^{s-k}\,(y\eta\,\bar y\bar\eta)^k\,(\eta\xi\,\bar\eta\bar\xi)^l \times C^{(1)}(\xi)\,C^{(1)}(\eta)\,. \tag{7.44}$$

We can then calculate the effect of this redefinition by first evaluating

$$\delta J = D(\delta\omega_0^{(2)})\,, \tag{7.45}$$

and then determining the corresponding Fronsdal current thereof by the methods presented in Section 6.10. This allows us to determine the induced changes to the Fronsdal current. Using this information, one is then led to solve the following recursive relation in order to cancel the single successor (7.38)

$$\begin{aligned}
a_{L+1,s} &= 1 = k^{(s)}_{1,L}\, b^{(s)}_{L}\,, \\
a_{L,s} &= 0 = k^{(s)}_{1,L-1}\, b^{(s)}_{L-1} + k^{(s)}_{2,L}\, b^{(s)}_{L}\,, \\
a_{L-1,s} &= 0 = k^{(s)}_{1,L-2}\, b^{(s)}_{L-2} + k^{(s)}_{2,L-1}\, b^{(s)}_{L-1} + k^{(s)}_{3,L}\, b^{(s)}_{L}\,, \\
&\;\;\vdots \\
a_{1,s} &= 0 = k^{(s)}_{2,0}\, b^{(s)}_{0} + k^{(s)}_{3,1}\, b^{(s)}_{1}\,,
\end{aligned}$$

where the coefficients are given by

$$k^{(s)}_{1,l} = (l+1)^2\,, \tag{7.46a}$$

$$k^{(s)}_{2,l} = -(2(l+1)^2 + 2ls + s^2)\,, \tag{7.46b}$$

$$k^{(s)}_{3,l} = (l+s+1)^2\,. \tag{7.46c}$$

By solving this recursive system of equations, one can deduce the shift in the prefactor $\tilde{a}^{(s)}_0$ of the minimal interaction term (7.1), i.e.

$$\tilde{a}^{(s)}_0 \to \tilde{a}^{(s)}_0 + C^{(s)}_{L+1}\,. \tag{7.47}$$

The form of $C^{(s)}_l$ for arbitrary spin s was first derived in [62] (see also [63]) and is given by

$$C^{(s)}_l = \frac{(l+s+1)!\,{}_3F_2(1-s, 1-s, -2s; 2-2s, l-s+2; 1)}{2(2s-1)(s-1)!s!\Gamma(l-s+2)}\,. \tag{7.48}$$

For the case of $s=2$ it becomes

$$C^{(2)}_l = \frac{1}{12}(l+1)(l+2)^2(l+3)\,. \tag{7.49}$$

From our discussion in the last section, it is clear that this result allows us to easily project on the minimal interaction term (7.1) and to determine its coefficient:

$$\tilde{a}^{(s)}_0 = \sum_{l=0}^{\infty} a_{l,s}\, C^{(s)}_l\,. \tag{7.50}$$

However, there is a problem. To illustrate this, we focus on the case $s=2$. Using the explicit results for $a_{l,2}$ extracted from Vasiliev theory, one obtains

$$\tilde{a}^{(2)}_0 = \frac{1}{24}\sum_{l=0}^{\infty}(l+1)\,, \tag{7.51}$$

which obviously diverges. More generally, one can show that the expression (7.50) diverges for all spins $s \geq 2$. This can be seen as follows: for large values of l, one can extract the following asymptotic behavior [62]

$$C^{(s)}_l \sim l^{2s}\,, \tag{7.52}$$

which has to be compared with the coefficients $a_{l,s}$ extracted from Vasiliev equations

$$a_{l,s} \sim \frac{1}{l^3}\,. \tag{7.53}$$

We will discuss this divergence and possible interpretations in the next section.

We have only considered the minimal current sector so far. By analogous arguments, one can show that all other sectors can be completely removed using the algorithm outlined above.

In [62], this analysis was extended to the three-dimensional case. It is shown that for three-dimensional Vasiliev theory the coefficient of the minimal interaction $\tilde{a}_0^{(2)}$ diverges for all spins $s > 2$.

Furthermore, the authors perform a Witten diagram calculation in three-dimensional Vasiliev theory which allows them to determine the three-point function $\langle j_s j_0 j_0 \rangle$ of the boundary conformal field theory involving two spin-0 and a single spin-s current. They first determine this three-point function from the general pseudo-local interaction term[4] (7.36) with arbitrary coefficients $a_{l,s}$ and then show[5] that it leads to the same result as the minimal interaction term (7.37) with coefficient (7.50). This provides additional evidence for our construction as it is common lore in the AdS/CFT literature that physically allowed field redefinitions should leave boundary correlation functions invariant [64]. Note that this calculation does not rely on any specific proposal for the dual conformal field theory. It merely uses the fact that the three-point function calculated from Witten diagrams should stay invariant under physically allowed field redefinitions.

In fact in Section 2 of [64] - one of the seminal publications on correlators in AdS/CFT - it is shown that, using the toy model (7.8), the resulting boundary three point correlator agrees with the one calculated from the minimal interaction with modified coefficient.

7.3 DISCUSSION AND POSSIBLE INTERPRETATIONS

We have seen that the minimal interaction term extracted by our calculation diverges. In this calculation we made three crucial assumptions:

- Schwinger–Fock gauge (3.113).
- Second order Fronsdal fields are given by

$$\phi_{n(s)} \sim e_n^{(2)\,a(s-1)}(\bar{e}_{na})^{s-1}\,. \tag{7.54}$$

- Field theoretical methods are applicable.

We discuss these assumptions in more detail in the following: one can show easily that the linear equations of motion are independent of

4 Since three-dimensional Vasiliev theory contains a complex instead of a real scalar field, all interaction terms contain a pair of complex conjugated scalar fields $\Phi^\dagger \ldots \Phi$ instead of $\Phi \ldots \Phi$ in the four-dimensional expressions.

5 An important assumption in deriving this result is that the integral over AdS space commutes with the infinite sum over successors.

the gauge choice. Let us consider the three-dimensional Vasiliev theory although completely analogous arguments also hold for the four-dimensional case. Without imposing the Schwinger–Fock gauge the field $\mathcal{A}^{(1)}_\alpha$ contains an additional $\partial^z_\alpha \epsilon^{(1)}(y,z)$ piece as was discussed after (3.113). The Vasiliev equation (3.105c) determines the z-dependence of the masterfield $\mathcal{W}$ by

$$\partial^z_\alpha \mathcal{W}^{(1)} = D_\Omega(\partial^z_\alpha \epsilon^{(1)} + \dots) \,. \tag{7.55}$$

Here and in the following, an ellipses denotes terms that are also present in the Schwinger–Fock gauge. Since $\partial^z_\alpha D_\Omega = D_\Omega \partial^z_\alpha$, we see that the masterfield $\mathcal{W}^{(1)}$ is now given by

$$\mathcal{W}^{(1)} = D_\Omega \epsilon^{(1)} + \dots \,. \tag{7.56}$$

This gauge term will drop out of the dynamical equation, $D_\Omega \mathcal{W}^{(1)} = 0$, because of $D^2_\Omega = 0$. Furthermore, $\mathcal{B}^{(1)}$ is independent of $\mathcal{A}^{(1)}_\alpha$ and can therefore not be affected by a different gauge choice for $\mathcal{A}^{(1)}_\alpha$. Beyond the linear order however, the gauge term $\partial^z_\alpha \epsilon^{(1)}$ will not drop out anymore. For example at the second order, the z-dependence of $\mathcal{W}^{(2)}$ is determined by

$$\partial^z_\alpha \mathcal{W}^{(2)} = D_\Omega(\partial^z_\alpha \epsilon^{(2)}) + [\mathcal{W}^{(1)}, \partial^z_\alpha \epsilon^{(1)}]_\star + \dots \,. \tag{7.57}$$

The first term will drop out of the dynamical equation $D_\Omega \mathcal{W}^{(2)}$ by the same mechanism as at linear order. However, the second term will generically lead to additional terms in the equations of motion for the twisted and physical one-form. Since $\mathcal{A}_\alpha$ is an auxiliary field (it encodes the coupling between $\mathcal{W}$ and $\mathcal{B}$ in Vasiliev equations), it should not contain its own degrees of freedom and therefore the spacetime zero-form ϵ has to be a functional of C, i.e.

$$\epsilon = \epsilon[C] \,. \tag{7.58}$$

The equations of motion obtained in such a way are related to the ones derived from Schwinger–Fock gauge by pseudo-local field redefinitions which may or may not be physically permissible. The Schwinger–Fock gauge is appealing since we know how to extract manifestly Lorentz covariant equations of motion from it as we discussed in Section 5.2. However, a possible interpretation of our results may be that the Schwinger–Fock gauge needs to be modified.

The identification of the Fronsdal field (7.54) is of course by no means the only possibility to construct a tensor with the symmetry properties and transformation behavior of the Fronsdal fields out of (generalized) vielbeins. Any other identification will be related to (7.54) by a field redefinition. If the reason of the divergent coefficient was our identification of the Fronsdal field, this would imply that the field redefinition obviously changes this coefficient (in order to make it finite). But our class of field redefinitions cannot do so by construction. Therefore, any identification that is related to the definition (7.54) by an allowed field redefinition will also lead to a divergent coefficient. Our results could thus indicate that the identification (7.54) may require some modification outside of this class of field redefinitions but so far we lack a concrete proposal of how to do this.

Our methods assume that source terms extracted from Vasiliev theory can be analyzed using field theoretical methods. More precisely, we interpreted the pseudo-local interactions of Vasiliev theory as an infinite expansion in successors. Each of these successors can then be individually removed using a local field redefinition in the standard field theoretical sense. This modifies the coefficient of the minimal interaction term. We then summed over all contributions of the infinite number of successors. Similarly, the authors of [62] calculated the boundary three-point correlator $\langle j_s j_0 j_0 \rangle$ using a Witten diagram calculation by commuting the infinite sum of successors with the integral over AdS space. A possible interpretation of our results is that this field-theoretical procedure is not applicable to Vasiliev theory. We will comment on this point further in the conclusions.

Finally, one may try to regularize the coefficient $a_s = \sum_{l=0}^{\infty} a_{l,s} C_l^{(s)}$ in front of the minimal interaction term (7.1). We performed some preliminary studies in this direction using ζ-function regularization and other methods based on analytic continuation in the spin s. However so far, we have not found a regularization scheme which systematically works for all spins and therefore this issue requires further study. If an appropriate regularization scheme was to be found, one would want to find a physical interpretation of it. In fact, using regularization techniques the authors of [65] were able to calculate boundary three-point functions from *the zero-form sector* of four-dimensional Vasiliev theory. This sector also contains pseudo-local interaction terms in four dimensions. However, their methods do not have an obvious generalization to the one-form sector.

8 CONCLUSION AND OUTLOOK

Christian Fronsdal's seminal paper [29] on the free theory of higher-spin fields closes by posing the question whether there exists a non-linear theory of interacting higher-spin fields. This question has become known in the literature as the "Fronsdal problem".

Different approaches have been taken to solve this problem. These include the Noether procedure [66–77] and the holographic reconstruction of higher-spin theories [78–82]. Encouraging progress has been achieved over the last years. For example, the complete cubic action for arbitrary dimension was obtained in [82] combining techniques based on the Noether procedure and holographic reconstruction. This progress was partly based on a similar result in our publication [26] for the three-dimensional theory which we did not report in this thesis. Recently, even quartic couplings have been studied [78, 79].

Despite these encouraging results, the only consistent non-linear theory which reproduces the Fronsdal equations upon linearization is currently given by Vasiliev equations. These equations are formulated in a highly non-standard way. As a result, it is remarkably challenging to make a connection with the Fronsdal formalism and thereby extract physics out of them. In this respect, the alternative approaches to Vasiliev theory listed above provide us with a clearer physical picture of the underlying theory. The present thesis confronted this challenge.

In three dimensions, for the case of vanishing scalar field (and twisted fields), most of these difficulties can be avoided and the dynamics of the theory is described by a Chern–Simons action. In Chapter 4, we successfully developed an efficient algorithm to translate this theory in the metric-like language of the Fronsdal fields. While the resulting expressions are involved, our algorithm allows us in principle to straightforwardly extract corrections to the Fronsdal equation perturbatively in the higher-spin fields.

For the case of non-vanishing scalar field, the only known description of the theory is given in terms of Vasiliev equations. In Chapter 5 and 6, we obtained the equations of motion for higher-spin and matter fields up to second order in perturbations of an AdS background. This required the resolution of considerable technical and conceptual difficulties. The former are mainly due to the infinite number of auxiliary variables of Vasiliev theory while the latter arise from the need for manifest local Lorentz covariance and the locality properties of field redefinitions. Both in three and four dimensions, our work represents the first systematic study of the interactions of Vasiliev theory expressed in terms of physical fields only.

Our analysis sheds light on the spectrum of three-dimensional Vasiliev theory. We have shown that its twisted sector can be set to zero consistently (at least up to second order in perturbations). This is important as these fields have no immediate interpretation within the Gaberdiel–Gopakumar duality.

The extracted equations of motion for both three and four dimensional Vasiliev theory share the common feature that their interaction terms are of pseudo-local form, i.e. they contain infinitely many derivatives at fixed spin. These terms are therefore potentially non-local. As higher-spin theories are thought to be related to the tensionless limit of string theory, there is no reason to expect that Vasiliev theory is a local field theory. This is however very different for the order we are considering: it has been established by Metsaev [57] that second order corrections to the Fronsdal equation only contain a finite number of derivatives and their structure is completely known.

In Chapter 7, we attempted to extract the free coefficients of this minimal Metsaev structure from Vasiliev theory. As we discussed in detail, the pseudo-local interactions extracted from Vasiliev equations do not necessarily stand in contradiction to Metsaev's results, as the infinite sum of successors could merely encode a certain coefficient of the minimal interaction term. Such a complicated encoding might be necessary in order to formulate the theory in unfolded language. In fact, it was widely expected that Vasiliev theory would lead to pseudo-local interactions but this was not considered a problem for precisely the reasons outlined above. We then brought the interaction terms of Vasiliev theory in minimal Metsaev form using only physically allowed field redefinitions. Surprisingly, the resulting coefficients are divergent and the dependence on the parameter θ is not of the expected form. It is conceivable that these two problems are related. Note that the possibility of divergent coefficients was first mentioned in [1, 83] and is consistent with related divergences in the zero-form sector observed in [65, 84].

Taken at face value, this suggests that the interactions extracted from Vasiliev theory are non-local. This would be worrisome, as it stands in contradiction to Metsaev's results [57], holographic reconstructions of the cubic couplings [80–82] and considering a theorem by Barnich and Henneaux which shows that for an arbitrary gauge theory there are no obstructions to the Noether procedure if one allows for non-local interaction terms [69].

However, our results may also suggest that the recipe of obtaining physical equations of motion from Vasiliev equations needs to be modified. It is important to stress that this process relies on certain assumptions which we discussed in detail in Section 7.3. Any of these may require some modification in light of our results and deserve further study. While this thesis was being completed, Vasiliev published a paper [85] in which he analyses the zero-form sector of four-dimensional

Vasiliev theory at second order. This sector also contains pseudo-local interaction terms which can be brought in minimal form - similar to our discussion in Chapter 7. He then makes a concrete proposal for such a field redefinition. While we have only suggested a criterion for the allowed class of field redefinitions in the one-form sector, it seems likely that Vasiliev's field redefinition is outside our class when appropriately generalized to the zero-form sector. Using the formalism outlined in this thesis, one could study the interaction terms in the redefined zero-form sector. One could then check whether the correct coefficients of the Metsaev minimal structure are reproduced. If this is so, one might take the point of view that this field redefinition is part of the recipe to extract equations of motion from Vasiliev theory. In this case, it would be important to find an appropriate generalization of this redefinition for the one-form sector and higher orders in perturbation theory.

Vasiliev also conjectures this field redefinition to be part of a class which leaves boundary correlation functions invariant that are obtained in a non-standard way. It is important to stress that this conjecture does not stand in any contradiction to our results as the correlation functions are to be obtained by extending Vasiliev theory by master-fields of higher form degree [86]. One then interprets a closed spacetime four-form of this extended theory as a generating functional for boundary correlation functions in the spirit of an on-shell Lagrangian. One would need to solve this extended Vasiliev system up to third order in perturbations in order to extract boundary three-point functions. It is therefore not surprising that so far not even the boundary two-point functions have been obtained using this method (see however [87, 88] for an interesting new approach).

In the author's opinion, any significant progress in the understanding of the interactions of Vasiliev theory would need to pass the following benchmark tests: the three-point boundary correlators have to be obtained directly from Vasiliev equations in both the zero and one-form sector by a procedure that has an obvious generalization to higher orders. A criterion for the allowed class of field redefinitions has to be given. Ideally, such a method should not rely on regularization schemes or provide a physical justification for them. We hope that this progress will be achieved in the near future.

Part IV

APPENDICES

A

CONVENTIONS

This appendix summarizes the conventions used throughout this thesis.

A.1 SYMMETRIZATION

Indices on the same level and denoted by the same letter are to be symmetrized by adding all necessary permutations, e. g. $X_\alpha Y_\alpha$ is understood as $X_{\alpha_1} Y_{\alpha_2} + X_{\alpha_1} Y_{\alpha_2}$, without further normalization.

A symmetric rank-n tensor will be denoted as $T_{\alpha(n)}$, which means the tensor components $T_{\alpha_1 \dots \alpha_n}$ are completely symmetric with respect to exchange of any two indices, e. g. $X_\alpha Y_{\alpha(n-1)}$ should be understood as $X_{\alpha_1} Y_{\alpha_2 \dots \alpha_n} +$ $(n-1)$ terms.

In exceptional cases it is also useful to use brackets to denote symmetrization, i.e. $y_{(\alpha_1} y_{\alpha_2} \dots y_{\alpha_n)} = y_{\alpha_1} y_{\alpha_2} \dots y_{\alpha_n} +$ permutations.

A.2 GENERAL RELATIVITY

We use the mostly plus convention $(-,+,+,\dots,+)$. Spacetime indices are denoted by $m, n, o, p \dots$ whereas $a, b, c \dots$ stand for local Lorentz indices. The Ricci tensor is defined by

$$R_{mn} := R_{mrn}{}^{r} \,. \tag{A.1}$$

The AdS$_d$ radius l is related to the cosmological constant Λ by

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2} \,. \tag{A.2}$$

A.3 METRIC-LIKE THEORY

The Killing form is defined to be one half of the matrix trace in the fundamental representation of $\mathfrak{sl}(3,\mathbb{R})$,

$$\kappa_{\mathcal{AB}} = \frac{1}{2} \operatorname{tr}\left(J_{\mathcal{A}} J_{\mathcal{B}}\right) \,, \tag{A.3}$$

and therefore

$$\kappa_{ab} = \eta_{ab} \,, \tag{A.4a}$$

$$\kappa_{aB} = 0 \,. \tag{A.4b}$$

The anti-symmetric and symmetric structure constants are given by

$$f_{\mathcal{ABC}} = \frac{1}{2} \operatorname{tr}\left([J_{\mathcal{A}}, J_{\mathcal{B}}] J_{\mathcal{C}}\right) \,, \tag{A.5a}$$

$$d_{\mathcal{ABC}} = \frac{1}{2} \operatorname{tr}\left(\{J_{\mathcal{A}}, J_{\mathcal{B}}\} J_{\mathcal{C}}\right) \,, \tag{A.5b}$$

such that

$$f_{Abc} = f_{ABC} = 0 \,, \tag{A.6a}$$
$$f_{abc} = \epsilon_{abc} \,, \tag{A.6b}$$
$$d_{abc} = d_{ABc} = 0 \,. \tag{A.6c}$$

The structure constants obey the following useful identities

$$d_{Abc}\,\kappa^{bc} \;=\; 0 \,, \tag{A.7a}$$
$$d_{Abc}\,d^{A}{}_{de} \;=\; -\tfrac{2}{3}\,\kappa_{bc}\,\kappa_{de} + 2\,\kappa_{d(b}\,\kappa_{c)e} \,. \tag{A.7b}$$

A.4 THREE-DIMENSIONAL VASILIEV THEORY

A.4.1 *Spinorial Indices*

Spinorial indices are denoted by Greek letters. We raise and lower these indices as follows

$$y^{\alpha} = \epsilon^{\alpha\beta} y_{\beta} \qquad\qquad y_{\alpha} = y^{\beta}\epsilon_{\beta\alpha} \,, \tag{A.8}$$

where $\epsilon_{01} = \epsilon^{01} = 1$. This implies that $\epsilon^{\alpha\beta}\epsilon_{\gamma\beta} = \delta^{\alpha}_{\gamma}$.

A.4.2 *Star Products*

Using the integral formula

$$(f \star g)(y,z) = \tfrac{1}{(2\pi)^2} \int \mathrm{d}^2u\, \mathrm{d}^2v \; f(y+u, z+u) \times g(y+v, z-v) \exp\left(i v^{\alpha} u_{\alpha}\right) . \tag{A.9}$$

one can easily derive

$$y_{\alpha} \star f = (y_{\alpha} + i\partial^{y}_{\alpha} - i\partial^{z}_{\alpha}) f \,, \tag{A.10a}$$
$$z_{\alpha} \star f = (z_{\alpha} + i\partial^{y}_{\alpha} - i\partial^{z}_{\alpha}) f \,, \tag{A.10b}$$
$$f \star y_{\alpha} = (y_{\alpha} - i\partial^{y}_{\alpha} - i\partial^{z}_{\alpha}) f \,, \tag{A.10c}$$
$$f \star z_{\alpha} = (z_{\alpha} + i\partial^{y}_{\alpha} + i\partial^{z}_{\alpha}) f \,, \tag{A.10d}$$

from these basic identities it follows straightforwardly that

$$[y_{\alpha}, f]_{\star} = 2i\, \partial^{y}_{\alpha} f \,, \tag{A.11a}$$
$$[z_{\alpha}, f]_{\star} = -2i\, \partial^{z}_{\alpha} f \,, \tag{A.11b}$$
$$\{y_{\alpha}, f\}_{\star} = 2\,(y_{\alpha} - i\partial^{z}_{\alpha})\, f \,, \tag{A.11c}$$
$$\{z_{\alpha}, f\}_{\star} = 2\,(z_{\alpha} + i\partial^{y}_{\alpha})\, f \,, \tag{A.11d}$$

which in turn implies

$$[L^{y}_{\alpha\alpha}, f]_{\star} = (y_{\alpha} - i\partial^{z}_{\alpha})\partial^{y}_{\alpha} f \,, \tag{A.12a}$$
$$\{L^{y}_{\alpha\alpha}, f\}_{\star} = -i(y_{\alpha} - i\partial^{z}_{\alpha})(y_{\alpha} - i\partial^{z}_{\alpha}) f + i\partial^{y}_{\alpha}\partial^{y}_{\alpha} f \,, \tag{A.12b}$$
$$[L^{z}_{\alpha\alpha}, f]_{\star} = (z_{\alpha} + i\partial^{y}_{\alpha})\partial^{z}_{\alpha} f \,. \tag{A.12c}$$

These identities also imply for $P_{\alpha\alpha} = \phi L_{\alpha\alpha}$ that

$$[P_{\alpha\alpha}, f]_\star = \phi\,(y_\alpha - i\partial^z_\alpha)\partial^y_\alpha\, f\,, \tag{A.13a}$$
$$[P_{\alpha\alpha}, f\,\psi]_\star = \phi\, i\partial^y_\alpha\partial^y_\alpha f\psi - i\,\phi\,(y_\alpha - i\partial^z_\alpha)(y_\alpha - i\partial^z_\alpha) f\,\psi\,. \tag{A.13b}$$

A.4.3 *Fourier Space*

We define the Fourier transformed fields as follows

$$F(y,\phi) = \int \mathrm{d}^2\xi\, e^{iy\xi}\, F(\xi,\phi)\,, \tag{A.14}$$

and analogously for all other fields. The source terms involving two first order physical zero-forms are then certain q-forms of the type

$$J_q = \int \mathrm{d}^2\xi\,\mathrm{d}^2\eta\, K_q(\xi,\eta,y)\,\hat{C}^{(1)}(\xi,\phi|x)\,\hat{C}^{(1)}(\eta,-\phi|x)\,, \tag{A.15}$$

where the relative sign in ϕ between the two zero-forms was explained in (5.32). The kernel K_q is given for the various form-degrees by

$$K_0 = K(\xi,\eta,y)\,, \qquad K_1 = \bar{e}^{\alpha\alpha}K_{\alpha\alpha}(\xi,\eta,y)\,, \tag{A.16a}$$
$$K_2 = E^{\alpha\alpha}J_{\alpha\alpha}(\xi,\eta,y)\,, \qquad K_3 = E\,J(\xi,\eta,y)\,, \tag{A.16b}$$

and we have used the definitions

$$E^{\alpha\alpha} := \bar{e}^{\alpha}{}_{\beta}\wedge\bar{e}^{\alpha\beta}\,, \qquad E := E_{\alpha\alpha}\wedge\bar{e}^{\alpha\alpha}\,, \tag{A.17}$$

which obey the following identities

$$\bar{e}^{\alpha\alpha}\wedge\bar{e}^{\beta\beta} = \frac{1}{4}\epsilon^{\alpha\beta}E^{\alpha\beta}\,, \tag{A.18a}$$
$$E^{\alpha\beta}\wedge\bar{e}^{\gamma\delta} = \frac{1}{6}(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\beta\gamma}\epsilon^{\alpha\delta})E\,. \tag{A.18b}$$

With the help of the equations of motion for the Fourier-transformed fields,

$$\nabla\hat{C}^{(1)}(\xi,+\phi) = -\frac{i}{2}\,\phi\,\bar{e}^{\alpha\alpha}\left(\xi_\alpha\xi_\alpha - \partial^\xi_\alpha\partial^\xi_\alpha\right)\hat{C}^{(1)}(\xi,+\phi)\,, \tag{A.19a}$$
$$\nabla\hat{C}^{(1)}(\eta,-\phi) = +\frac{i}{2}\,\phi\,\bar{e}^{\alpha\alpha}\left(\eta_\alpha\eta_\alpha - \partial^\eta_\alpha\partial^\eta_\alpha\right)\hat{C}^{(1)}(\eta,-\phi)\,, \tag{A.19b}$$

and of the identities (A.18), we find the following Fourier representations for the adjoint derivative D:

$$DK(\xi,\eta,y) = \bar{e}^{\alpha\alpha}\,O_{\alpha\alpha}\,K(\xi,\eta,y)\,, \tag{A.20a}$$
$$D\bar{e}^{\alpha\alpha}K_{\alpha\alpha}(\xi,\eta,y) = \frac{1}{4}\,E^{\alpha\alpha}\,O_{\alpha\nu}\,K_\alpha{}^\nu(\xi,\eta,y)\,, \tag{A.20b}$$
$$DE^{\alpha\alpha}J_{\alpha\alpha}(\xi,\eta,y) = \frac{1}{6}\,E\,O^{\alpha\alpha}\,J_{\alpha\alpha}(\xi,\eta,y)\,, \tag{A.20c}$$

where we have defined

$$O_{\alpha\alpha} := \frac{i\phi}{2}\left[(\eta_\alpha\eta_\alpha - \partial^\eta_\alpha\partial^\eta_\alpha) - \left(\xi_\alpha\xi_\alpha - \partial^\xi_\alpha\partial^\xi_\alpha\right) + 2iy_\alpha\partial^y_\alpha\right]\,. \tag{A.20d}$$

We will also need the action of the twisted-adjoint covariant derivative $\tilde{D}$ acting on q-forms linear in $\hat{C}^{(1)}$ and $\hat{\omega}^{(1)}$:

$$J_q = \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta \,\Big\{ L_q(\xi,\eta,y)\, \hat{C}^{(1)}(\xi,\phi)\, \hat{\omega}^{(1)}(\eta,-\phi) \\ + \bar{L}_q(\xi,\eta,y)\, \hat{\omega}^{(1)}(\xi,\phi)\, \hat{C}^{(1)}(\eta,\phi) \Big\}\,, \tag{A.21}$$

where the kernels L_q and $\bar{L}_q$ are given by

$$L_1 = L(\xi,\eta,y)\,, \qquad \bar{L}_1 = \bar{L}(\xi,\eta,y)\,, \tag{A.22a}$$
$$L_2 = \bar{e}^{\alpha\alpha} L_{\alpha\alpha}(\xi,\eta,y)\,, \qquad \bar{L}_2 = \bar{e}^{\alpha\alpha} \bar{L}_{\alpha\alpha}(\xi,\eta,y)\,, \tag{A.22b}$$
$$L_3 = E^{\alpha\alpha} S_{\alpha\alpha}(\xi,\eta,y)\,, \qquad \bar{L}_3 = E^{\alpha\alpha} \bar{S}_{\alpha\alpha}(\xi,\eta,y)\,, \tag{A.22c}$$

Using the equations of motion for $\hat{\omega}^{(1)}$ and $\hat{C}^{(1)}$, we again obtain a Fourier representation for the twisted adjoint covariant derivative $\tilde{D}$:

$$\tilde{D}L(\xi,\eta,y) = \bar{e}^{\alpha\alpha}\, I_{\alpha\alpha}\, L(\xi,\eta,y)\,, \tag{A.23}$$
$$\tilde{D}h^{\alpha\alpha} L_{\alpha\alpha}(\xi,\eta,y) = \frac{1}{4} E^{\alpha\alpha}\, I_{\alpha\nu}\, L_{\alpha}{}^{\nu}(\xi,\eta,y)\,, \tag{A.24}$$

where we have defined

$$I_{\alpha\alpha} := \frac{i\phi}{2}\Big[(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha) - \Big(\xi_\alpha\xi_\alpha - \partial^\xi_\alpha\partial^\xi_\alpha\Big) + 2i\,\eta_\alpha \partial^\eta_\alpha \Big]\,. \tag{A.25}$$

Analogous expressions hold for the barred kernels upon replacing $I_{\alpha\alpha}$ with $\bar{I}_{\alpha\alpha}$ defined as

$$\bar{I}_{\alpha\alpha} := \frac{i\phi}{2}\Big[(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha) - (\eta_\alpha\eta_\alpha - \partial^\eta_\alpha\partial^\eta_\alpha) + 2i\,\xi_\alpha \partial^\xi_\alpha \Big]\,. \tag{A.26}$$

Furthermore, we will be interested in situations where the twisted adjoint covariant derivative $\tilde{D}$ acts on q-forms linear in $\hat{C}^{(1)}$ which are of the form

$$J_q = \int \mathrm{d}^2\xi\, K_q(\xi,y)\, \hat{C}^{(1)}(\xi,\phi|x)\,, \tag{A.27}$$

The kernel K_q is given for the various form-degrees by

$$K_0 = K(\xi,y)\,, \qquad K_1 = \bar{e}^{\alpha\alpha} K_{\alpha\alpha}(\xi,y)\,, \tag{A.28a}$$
$$K_2 = E^{\alpha\alpha} J_{\alpha\alpha}(\xi,y)\,, \qquad K_3 = E\, J(\xi,y)\,, \tag{A.28b}$$

The twisted adjoint covariant derivative $\tilde{D}$ then acts by

$$\tilde{D}K(\xi,\eta,y) = \bar{e}^{\alpha\alpha}\, O_{\alpha\alpha}\, K(\xi,\eta,y)\,, \tag{A.29a}$$
$$\tilde{D}\bar{e}^{\alpha\alpha} K_{\alpha\alpha}(\xi,\eta,y) = \frac{1}{4} E^{\alpha\alpha}\, O_{\alpha\nu}\, K_{\alpha}{}^{\nu}(\xi,\eta,y)\,, \tag{A.29b}$$
$$\tilde{D}E^{\alpha\alpha} J_{\alpha\alpha}(\xi,\eta,y) = \frac{1}{6} E\, O^{\alpha\alpha}\, J_{\alpha\alpha}(\xi,\eta,y), \tag{A.29c}$$

where we defined

$$O_{\alpha\alpha} := \frac{i\phi}{2}\Big[(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha) - \Big(\xi_\alpha\xi_\alpha - \partial^\xi_\alpha\partial^\xi_\alpha\Big) \Big]\,. \tag{A.29d}$$

A.5 FOUR-DIMENSIONAL VASILIEV THEORY

A.5.1 *Spinorial Indices*

The commuting oscillators y_α and $\bar{y}_{\dot{\alpha}}$ are conveniently combined

$$Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})\,. \tag{A.30}$$

Indices of Y^A can then be lowered or raised by

$$\epsilon^{AB} := \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{A.31}$$

Both y_α and $\bar{y}_{\dot{\alpha}}$ follow the same conventions for raising and lowering as in three dimensions.

A.5.2 *Star Products*

The star product is defined by

$$\begin{aligned}(f \star g)(Y, Z) = \tfrac{1}{(2\pi)^4} \int \mathrm{d}^4U\, \mathrm{d}^4V\ & f(Y+U, Z+U) \\ & \times g(Y+V, Z-V) \exp\left(iV^A U_A\right).\end{aligned} \tag{A.32}$$

Star products either involving only purely y or $\bar{y}$-dependent functions obey the same identities as in three dimensions. The translation generators contain both barred and unbarred indices and act as

$$[P_{\alpha\dot{\alpha}}, \bullet]_\star = \{(y_\alpha - i\partial^z_\alpha)\partial^y_{\dot{\alpha}} + (\bar{y}_{\dot{\alpha}} - i\partial^z_{\dot{\alpha}})\partial^y_\alpha)\}\,\bullet\,, \tag{A.33a}$$

$$\{P_{\alpha\dot{\alpha}}, \bullet\}_\star = -i\,\{(y_\alpha - i\partial^z_\alpha)(\bar{y}_{\dot{\alpha}} - i\partial^z_{\dot{\alpha}}) - \partial^y_\alpha\partial^y_{\dot{\alpha}}\}\,\bullet\ . \tag{A.33b}$$

A.5.3 *Fourier Transformations*

We use the following conventions for the Fourier transformation

$$f(Y) = \int \mathrm{d}^4\xi\, f(\xi) e^{iY^A\xi_A}\,, \tag{A.34}$$

with wave-vector $\xi^A = (\xi^\alpha, \bar{\xi}^{\dot{\alpha}})$.

B TECHNICALITIES

B.1 SOURCE TERMS

In this section, we collect the results for the various source terms of three-dimensional Vasiliev theory. We start by discussing the source terms in the physical sector before presenting the twisted sector.

B.1.1 *Physical Sector*

We first define the following exponentials for future convenience

$$Q = \exp i\left(tq(\eta + y)(y + \xi)\right) , \tag{B.1a}$$
$$P = \exp i\left(t(\eta + y)(y + \xi)\right) , \tag{B.1b}$$
$$K = \exp i(y - q\eta)(y + t\xi) , \tag{B.1c}$$
$$R^1 = \exp i\left(t(y - q(\eta + y))\xi\right) , \tag{B.1d}$$
$$R^2 = \exp i\left(q(y - t(\xi + y))\eta\right) , \tag{B.1e}$$
$$K_q = \exp iq(y\eta) = K|_{t=0} , \tag{B.1f}$$
$$K_t = \exp it(y\xi) = K|_{q=0} . \tag{B.1g}$$

The source term $\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})$ defined in (5.26d) depends on the free parameter g_0 of the field redefinition (3.123). We will first give the source term $\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C})$ for $g_0 = 1$ as this choice leads to a relatively concise form. We will then explain how one can easily obtain its corresponding form for $g_0 = 0$ which allows truncation to physical fields only.

The explicit form of this source term is

$$\mathcal{V}(\Omega, \Omega, \hat{C}, \hat{C}) = E^{\alpha\alpha} \int \mathrm{d}t\, \mathrm{d}q\, \mathrm{d}^2\xi\, \mathrm{d}^2\eta$$
$$\times \left(J^Q_{\alpha\alpha} + J^P_{\alpha\alpha} + J^K_{\alpha\alpha} + J^{R_1}_{\alpha\alpha} + J^{R_2}_{\alpha\alpha}\right) \hat{C}^{(1)}(\xi, \phi)\hat{C}^{(1)}(\eta, -\phi) .$$

The redefinition (3.123) only contributes to the last three kernels. For the choice $g_0 = 1$ the kernels are then given by

$$\begin{aligned}
J^Q_{\alpha\alpha} =& \{d_1\, y_\alpha y_\alpha + d_2\, \xi_\alpha\xi_\alpha + d_3\, \eta_\alpha\eta_\alpha + d_4\, \xi_\alpha\eta_\alpha + d_5\, \xi_\alpha y_\alpha \\
&+ d_6\, \eta_\alpha y_\alpha - d_7\left(y_\alpha\eta_\alpha(y\eta) - \xi_\alpha y_\alpha(y\xi) - \xi_\alpha\eta_\alpha(\xi\eta)\right)\}Q , \\
J^K_{\alpha\alpha} =& -\tfrac{1}{8} i(1-q)(1+t)\Big\{(q+t)(\eta_\alpha\eta_\alpha - \xi_\alpha\xi_\alpha) \\
&- (q+1)(t+1)y_\alpha\eta_\alpha - (q-1)(t-1)y_\alpha\xi_\alpha + (1+qt)y_\alpha y_\alpha \\
&- (q-1)(t+1)(2qt - q + t)\, \xi_\alpha\eta_\alpha\Big\}K \\
&- \tfrac{1}{16}(q-1)^3(q+1)(t-1)(t+1)^3\, (\eta\xi)\, \xi_\alpha\eta_\alpha\, K ,
\end{aligned}$$

$$
\begin{aligned}
J^{P}_{\alpha\alpha} &= \{p_1\,\xi_\alpha\xi_\alpha + p_2\,\eta_\alpha\eta_\alpha + p_3\,(\,t\,\xi_\alpha\eta_\alpha + \xi_\alpha y_\alpha + \eta_\alpha y_\alpha)\}P\,,\\
J^{R^1}_{\alpha\alpha} &= \{\rho_1\,\xi_\alpha\xi_\alpha + \rho_2\,\eta_\alpha\eta_\alpha + \rho_3\,y_\alpha\xi_\alpha + \rho_4\,y_\alpha\eta_\alpha + \rho_5\,\xi_\alpha\eta_\alpha\}R^1\\
&\quad + \tfrac{i}{16}\,(t^2-1)\,\{\xi_\alpha\eta_\alpha\,(t+2) - y_\alpha\eta_\alpha\}K_t\,,\\
J^{R^2} &= -\,J^{R^1}\begin{pmatrix} {\scriptstyle t\to -q} & {\scriptstyle R^1\to R^2}\\ {\scriptstyle q\to t}\;, & {\scriptstyle K_t\to K_q}\\ {\scriptstyle \xi\leftrightarrow\eta} & \end{pmatrix}\,,
\end{aligned}
$$

where we have used the following coefficients

$$
\begin{aligned}
d_1 &= \tfrac{i}{8}(-q+4q^2-3q^3+4qt-9q^2t+4q^3t+8q^2t^2+q^3t^2)\,,\\
d_2 &= -\tfrac{i}{8}(-3q+3q^3+4qt+q^2t-8q^3t+3q^3t^2)\,,\\
d_3 &= -\tfrac{i}{4}(-q+2qt+q^2t)\,,\\
d_4 &= \tfrac{i}{4}(3q-2q^2-2qt-3q^2t-2q^3t+10q^2t^2+2q^3t^2)\,,\\
d_5 &= -\tfrac{i}{4}(-2q^2+3q^3-2qt+2q^2t-6q^3t+2q^2t^2+q^3t^2)\,,\\
d_6 &= \tfrac{i}{4}(q-2q^2+2qt+3q^2t-2q^3t-2q^2t^2+2q^3t^2)\,,\\
d_7 &= \tfrac{1}{4}(-qt+2q^2t-2q^2t^2-2q^3t^2+3q^3t^3)\,,\\
\rho_1 &= \tfrac{i}{4}t(-1+q)(1+q+t)\,,\\
\rho_2 &= -\tfrac{i}{4}(-1+q)q\,,\\
\rho_3 &= \tfrac{i}{4}t(-1+q)^2(1+q+t)\,,\\
\rho_4 &= -\tfrac{i}{4}(-1+q)^2\,,\\
\rho_5 &= \tfrac{i}{4}(-1+q)(-1+q^2t+qt(1+t))\,,\\
p_1 &= -\tfrac{i}{4}t(1-t)^2\,,\\
p_2 &= -\tfrac{i}{4}(-t+t^3)\,,\\
p_3 &= \tfrac{i}{2}(-t+t^2)\,.
\end{aligned}
$$

The corresponding source term for $g_0 = 0$ can straightforwardly be obtained from these results. The terms $J^{Q}_{\alpha\alpha}$ and $J^{P}_{\alpha\alpha}$ are unaffected by any change in g_0 as they do not depend on the redefinition (3.123). For the kernels $J^{R^1}_{\alpha\alpha}$ and $J^{R^2}_{\alpha\alpha}$, one performs the following anti-symmetrization

$$
J^{R^1}_{\alpha\alpha} \to \frac{1}{2}\left(J^{R^1}_{\alpha\alpha} - J^{R^1}_{\alpha\alpha}\Big|_{t\to -t}\right)\,,\quad J^{R^2}_{\alpha\alpha} \to \frac{1}{2}\left(J^{R^2}_{\alpha\alpha} - J^{R^2}_{\alpha\alpha}\Big|_{q\to -q}\right)\,,
$$

and for J^K one has to apply both t and q anti-symmetrization.

B.1.2 *Twisted Sector*

TWISTED ZERO-FORM: The source term $\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C})$ of (5.45) is given by

$$
\widetilde{\mathcal{V}}(\Omega,\hat{C},\hat{C}) = \phi\,\bar{e}^{\alpha\alpha}\int \mathrm{d}^2\xi\mathrm{d}^2\eta\,K_{\alpha\alpha}(\xi,\eta,y)\,\hat{C}^{(1)}(\xi,\phi)\,\hat{C}^{(1)}(\eta,-\phi)\,,
$$

where the kernel reads

$$\begin{aligned}\int_0^1 \mathrm{d}t \Big\{ &\frac{1}{2}\, e^{i(y(1-t)-t\eta)\xi}\, \xi_\alpha \Big((1-t^2)\,(\xi_\alpha-\eta_\alpha) + (1-t)^2\, y_\alpha \Big) \\ &-\frac{1}{2}\, e^{i(y(1-t)-t\xi)\eta}\, \eta_\alpha \Big((1-t^2)\,(\eta_\alpha+\xi_\alpha) - (1-t)^2\, y_\alpha \Big) \\ &+\frac{1}{4}\,(t^2-1)\, e^{i(y-\eta)(y+t\xi)} \\ &\qquad \times \left(g_0\,(y-\eta)_\alpha(y-\eta)_\alpha - 2(y-\eta)_\alpha\xi_\alpha + g_0\,\xi_\alpha\xi_\alpha\right) \\ &+\frac{1}{4}\,(t^2-1)\, e^{i(y+\xi)(t\eta-y)} \\ &\qquad \times \left(g_0\,(y+\xi)_\alpha(y+\xi)_\alpha - 2\,(y-\xi)_\alpha\eta_\alpha + g_0\,\eta_\alpha\eta_\alpha\right) \Big\}\,. \end{aligned}$$

We consider the bosonic theory and therefore $\hat{C}^{(1)}(\xi) = \hat{C}^{(1)}(-\xi)$.

TWISTED ONE-FORM: As discussed in Section 5.5.2, one obtains after performing the redefinition $M_1^{(2)}$:

$$\begin{aligned}\tilde{\mathcal{V}}'(\Omega,\hat{\omega},\hat{C}) = \bar{\epsilon}^{\alpha\alpha} \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta\, \Big\{ &L_{\alpha\alpha}(\xi,\eta,y)\, \hat{C}^{(1)}(\xi,\phi)\, \hat{\omega}^{(1)}(\eta,-\phi) \\ &+ \bar{L}_{\alpha\alpha}(\xi,\eta,y)\, \hat{\omega}^{(1)}(\xi,\phi)\, \hat{C}^{(1)}(\eta,\phi) \Big\}\,,\end{aligned}$$

where $\hat{\omega}^{(1)}(\xi) = \hat{\omega}^{(1)}(-\xi)$ and $\hat{C}^{(1)}(\xi) = \hat{C}^{(1)}(-\xi)$ since we are considering the bosonic theory. The kernels are given by

$$\begin{aligned} L_{\alpha\alpha} = \int_0^1 \mathrm{d}t\, \Big\{ &\frac{1}{4}(t^2-1)\,(y_\alpha-\eta_\alpha-\xi_\alpha)(y_\alpha-\eta_\alpha-\xi_\alpha)\, e^{i(y-\eta)(t\xi+\eta)} \\ &+\frac{1}{2}\eta_\alpha(y_\alpha-\xi_\alpha+(2t-1)\eta_\alpha)\, e^{i[(1-t)y-t\xi]\eta} \\ &-\frac{1}{4}(t^2-1)(y_\alpha-\xi_\alpha+\eta_\alpha)(y_\alpha-\xi_\alpha+\eta_\alpha) e^{i(ty+\eta)(ty+\xi)} \\ &+\frac{1}{4}\,(t^2-1)\,(1+g_0)\,(y_\alpha y_\alpha+\eta_\alpha\eta_\alpha+\xi_\alpha\xi_\alpha-2y_\alpha\eta_\alpha) \\ &\qquad \times \cos(t(y-\eta)\xi)\, e^{iy\eta} \\ &-\frac{1}{4}(t^2-1)\,(1+g_1)\,(y_\alpha y_\alpha+\xi_\alpha\xi_\alpha+\eta_\alpha\eta_\alpha-2\xi_\alpha\eta_\alpha) \\ &\qquad \times \cos(ty(\xi-\eta)) e^{i\eta\xi} \Big\}\,, \end{aligned}$$

and also by

$$\begin{aligned} \bar{L}_{\alpha\alpha} = \int_0^1 \mathrm{d}t\, \Big\{ &\frac{-1}{4}(t^2-1)(y_\alpha+\xi_\alpha-\eta_\alpha)(y_\alpha+\xi_\alpha-\eta_\alpha)\, e^{i(y+\xi)(t\eta-y)} \\ &+\frac{1}{2}\xi_\alpha(-y_\alpha+\eta_\alpha+(2t-1)\xi_\alpha)\, e^{i[(1-t)y-t\eta]\xi} \\ &+\frac{1}{4}(t^2-1)(y_\alpha-\xi_\alpha-\eta_\alpha)(y_\alpha-\xi_\alpha-\eta_\alpha) e^{i(ty-\eta)(ty+\xi)} \\ &-\frac{1}{4}(t^2-1)\,(1+g_0)\,(y_\alpha y_\alpha+\xi_\alpha\xi_\alpha+2y_\alpha\xi_\alpha+\eta_\alpha\eta_\alpha) \\ &\qquad \times \cos(t(y+\xi)\eta) e^{iy\xi} \\ &+\frac{1}{4}\,(t^2-1)\,(1+g_1)\,(y_\alpha y_\alpha+\xi_\alpha\xi_\alpha+\eta_\alpha\eta_\alpha+2\xi_\alpha\eta_\alpha) \\ &\qquad \times \cos(ty(\xi+\eta)) e^{i\xi\eta} \Big\}\,. \end{aligned}$$

B.2 OSCILLATOR REALIZATION

In this section, we will give a lightning review of the higher-spin algebra and explain its relation to the deformed oscillator algebra. The discussion will not be completely rigorous and we refer to [89] for a more complete account.

We will first briefly summarize various important concepts which will allow us to prove that the oscillator algebra is isomorphic to the (higher-spin) associative algebra $\mathcal{B}[\nu]$. In the main text this was summarized in Section 3.1.5.

B.2.1 *Necessary Concepts*

Definition. *An* associative algebra $\mathcal{A}$ *is a vector space with a bilinear map*

$$\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\,, \tag{B.2}$$

which is called product or multiplication and is associative

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \qquad \forall x, y, z \in \mathcal{A}\,. \tag{B.3}$$

The algebra $\mathcal{A}$ is unital if

$$\exists e \in \mathcal{A} : \quad e \bullet a = a \bullet e = a \qquad \forall a \in \mathcal{A}\,. \tag{B.4}$$

We will in the following only consider unital associative algebras and refer to them as associative algebras for sake of brevity.

Let us give two examples for associative algebras. Both will be of crucial importance for our later discussion.

Definition. *The free algebra $\mathbb{R} < T_i >$ of n indeterminants $T_1 \ldots T_n$ is spanned by the formal products*

$$T_{i_1} \ldots T_{i_k} \quad \textit{with} \quad i_l \in \{1, \ldots, n\} \quad \textit{and} \quad k = 0, 1, 2, \ldots \tag{B.5}$$

The multiplication rule is given by concatenation, i.e.

$$T_{i_1} \ldots T_{i_k} \bullet T_{j_1} \ldots T_{j_l} = T_{i_1} \ldots T_{i_k} T_{j_1} \ldots T_{j_l}\,. \tag{B.6}$$

It can be easily seen that this free algebra is indeed an associative algebra. Another example for an associative algebra which is less straightforward is given by the deformed oscillators discussed in Section 3.1.5.

Theorem. *The oscillator algebra $\mathcal{Y}[\nu]$ spanned by even symmetric products of deformed oscillators* (3.73)*,*

$$V_{\alpha(2n)} = \left(\frac{-i}{4}\right)^n P_- \hat{y}_{(\alpha_1} \ldots \hat{y}_{\alpha_{2n})} \qquad n = 0, 1, 2, \ldots\,, \tag{B.7}$$

forms an associative algebra with the multiplication given by the star product.

Proof. It is clear that the oscillator algebra is a vector space. Since the star product is obviously bilinear and associative, it remains to be shown that

$$V_{\alpha(2n)} \star V_{\beta(2m)} \in \mathcal{Y}[\nu]\,. \tag{B.8}$$

From

$$P_{-}\hat{y}_{\alpha} \star \hat{y}_{\beta} = \frac{1}{2}P_{-}\hat{y}_{(\alpha} \star \hat{y}_{\beta)} + i\,\epsilon_{\alpha\beta}\,\underbrace{P_{-}(1+\nu k)}_{P_{-}(1-\nu)} \tag{B.9}$$

we see that only the symmetric component of this "fundamental" star product contains $\hat{y}$-oscillators. By associativity it then follows that

$$V_{\alpha(2n)} \star V_{\beta(2m)} = \sum_{k=0}^{\min(2n,2m)} c_k\, V_{\alpha(2n-k)\beta(2m-k)}(\epsilon_{\alpha\beta})^k\,. \tag{B.10}$$

Note that from the arguments above, it follows that $c_k \in \mathbb{R}$ and therefore $\mathcal{Y}[\nu]$ also forms a Lie algebra over the real numbers. □

For later purposes, it is useful to introduce the notion of an ideal. One can then construct a quotient algebra which, roughly speaking, allows one to "set the elements of the ideal to zero".

Definition. *A subalgebra $I \subset \mathcal{A}$ is called a two-sided ideal if*

$$I \bullet \mathcal{A} \subset I \qquad \text{and} \qquad \mathcal{A} \bullet I \subset I\,. \tag{B.11}$$

Definition. *Given an ideal $I \subset \mathcal{A}$, we define the quotient algebra $\mathcal{A}/I$ as the associative algebra of equivalence classes $[a]$ with equivalence relation*

$$a \sim a + I\,. \tag{B.12}$$

Definition. *Given a set $X \subset \mathcal{A}$ we define $\langle X \rangle$, the ideal generated by X, as the smallest ideal containing all elements of X.*

An interesting feature of the associative algebras is that they can be turned into Lie algebras by the following theorem:

Theorem. *An associative algebra $\mathcal{A}$ forms a Lie algebra with respect to the Lie bracket*

$$[x,y] := x \bullet y - y \bullet x \qquad \forall x, y \in \mathcal{A}\,. \tag{B.13}$$

Proof. By definition $\mathcal{A}$ is a vector space. From associativity and bilinearity of the multiplication $\bullet$, it then follows that the Lie bracket fulfills the Jacobi identity and is bilinear and alternating. □

This leads to a natural question. Can we "turn this theorem around", i.e. can we construct an associative algebra from a Lie algebra $\mathfrak{g}$? This can be done by the following construction:

Definition. *The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ with basis t_i is defined by the quotient algebra*

$$U(\mathfrak{g}) = \frac{\mathbb{R} < t_i >}{\langle t_i t_j - t_j t_i - [t_i, t_j] \rangle} \,. \tag{B.14}$$

This definition seems to suggest that the universal enveloping algebra depends on the choice of basis. It is important to stress that this is not the case. There exists a more invariant definition in terms of the tensor algebra of $\mathfrak{g}$. We refer to Section 2 of [89] for more details.

We denote by $[t_i, t_i]$ the Lie bracket of two generators in $\mathfrak{g}$. At the level of the free algebra $\mathbb{R} < t_i >$ the commutator $t_i t_j - t_j t_i$ consists merely of formal products of generators. By considering the quotient algebra, one ensures that this formal commutator is in the same equivalence class as $f_{ij}{}^k\, t_k$, where $f_{ij}{}^k$ denote the structure constants of $\mathfrak{g}$ in this basis.

Furthermore, this equivalence relation ensures that the universal enveloping algebra is spanned by the equivalence classes containing

$$t_{(i_1} \dots t_{i_n)} \qquad\qquad n = 0, 1, \dots \tag{B.15}$$

since the equivalence relation identifies the antisymmetric product of two generators with linear combinations of single generators.

B.2.2 *Higher-Spin and Oscillator Algebra*

Using the concepts introduced in the last section, we will now show that the oscillator algebra indeed realizes the associative higher-spin algebra which is defined as follows:

Definition. *The higher-spin associative algebra $\mathcal{B}[\nu]$ is given by the quotient algebra*

$$\mathfrak{B}[\nu] = \frac{U(\mathfrak{sp}(2, \mathbb{R}))}{\langle C_2 + \frac{1}{4}(3 - 2\nu - \nu^2) \rangle} \,, \tag{B.16}$$

where C_2 denotes the quadratic Casimir of $\mathfrak{sp}(2, \mathbb{R})$ and $\nu \in \mathbb{R}$.

Similar as in our discussion for the universal enveloping algebra, it is clear that a basis for the higher-spin associative algebra is given by the equivalence classes containing

$$t_{(a_1} \dots t_{a_n)} - \text{traces} \,, \qquad\qquad n = 0, 1, \dots \,, \tag{B.17}$$

as the equivalence relation identifies the quadratic Casimir with a certain ν-dependent number. Traces are understood to be taken by contracting with the Killing form.

Note that from our discussion in Section 3.1.2, it follows that by defining $t_{\alpha\alpha} := \sigma^a_{\alpha\alpha} t_a$ one can rewrite (B.17) in spinorial notation as

$$t_{(\alpha_1 \alpha_2} \dots t_{\alpha_{2n-1} \alpha_{2n})} \,, \qquad\qquad n = 0, 1, \dots \,. \tag{B.18}$$

This obviously just corresponds to a change of basis of the Lie algebra $\mathfrak{sp}(2|\mathbb{R})$.

We are now in a position to present the following important theorem:

Theorem. *The higher-spin associative algebra is isomorphic to the oscillator algebra $\mathcal{Y}[\nu]$.*

Proof. We define the following linear map

$$\begin{array}{cccc} \varphi : & \mathcal{B}[\nu] & \to & \mathcal{Y}[\nu] \\ & t_{\alpha_1\alpha_2}\dots t_{\alpha_{2n-1}\alpha_{2n}} & \mapsto & P_- L_{\alpha_1\alpha_2}\star\dots\star L_{\alpha_{2n-1}\alpha_{2n}} \end{array}$$

where $L_{\alpha\beta}=\frac{i}{4}\hat{y}_{(\alpha}\star\hat{y}_{\beta)}$. For this map to be well-defined, it has to obey the following relations for $n,m\in\mathbb{N}$:

$$\varphi\left(t_{\gamma_1\gamma_2}\dots t_{\gamma_{2n-1}\gamma_{2n}}\left([t_{\alpha\alpha},t_{\beta\beta}]-\epsilon_{\alpha\beta}t_{\alpha\beta}\right)t_{\rho_1\rho_2}\dots t_{\rho_{2m-1}\rho_{2n}}\right)=0\,,$$
$$\varphi\left(t_{\gamma_1\gamma_2}\dots t_{\gamma_{2n-1}\gamma_{2n}}\left(-\tfrac{1}{2}t^{\alpha\alpha}t_{\alpha\alpha}-f(\nu)\right)t_{\rho_1\rho_2}\dots t_{\rho_{2m-1}\rho_{2m}}\right)=0\,,$$

where $f(\nu)=-\frac{1}{4}(3-2\nu-\nu^2)$. Since $L_{\alpha\alpha}P_-=P_-L_{\alpha\alpha}$ and $P_-^2=P_-$ it follows that

$$\varphi(t_{\alpha_1\alpha_2}\dots t_{\alpha_{2n-1}\alpha_{2n}})=\varphi(t_{\alpha_1\alpha_2})\star\dots\star\varphi(t_{\alpha_{2n-1}\alpha_{2n}})\,. \tag{B.19}$$

Therefore it is sufficient to check that

$$\varphi([t_{\alpha\alpha},t_{\beta\beta}]-\epsilon_{\alpha\beta}\,t_{\alpha\beta})=P_-\{[L_{\alpha\alpha},L_{\beta\beta}]-\epsilon_{\alpha\beta}\,L_{\alpha\beta}\}=0\,,$$
$$\varphi(-\tfrac{1}{2}t^{\alpha\alpha}\star t_{\alpha\alpha}-f(\nu))=P_-\{-\tfrac{1}{2}L^{\alpha\alpha}\star L_{\alpha\alpha}-f(\nu)\}=0\,.$$

The first equation follows from the fact that the subalgebra spanned by the generators $L_{\alpha\beta}$ realizes the $\mathfrak{sp}(2|\mathbb{R})$ algebra (see our discussion in Section 3.1.2). The second equation holds because of

$$-\frac{1}{2}P_-L^{\alpha\beta}\star L_{\alpha\beta}=-\frac{1}{4}P_-(3+2\nu k-\nu^2)=-\frac{1}{4}P_-(3-2\nu-\nu^2)\,, \tag{B.20}$$

where we have used $P_-k=-P_-$ and the result (3.76) for $L^{\alpha\beta}\star L_{\alpha\beta}$.

Each basis vector of $\mathcal{B}[\nu]$ is mapped to the corresponding basis vector in the oscillator algebra by

$$\varphi(\,t_{(\alpha_1\alpha_2}\dots t_{\alpha_{2n-1}\alpha_{2n})}\,)=P_-L_{(\alpha_1\alpha_2}\star\dots\star L_{\alpha_{2n-1}\alpha_{2n})}\,. \tag{B.21}$$

and therefore φ is bijective but by (B.19) it then provides us with an isomorphism between the two algebras. □

The higher-spin (Lie-)algebra $\mathfrak{hs}(\lambda)$ is then obtained by turning the associative higher-spin algebra $\mathfrak{B}[\nu]$ into a Lie algebra, as discussed above, and decomposing the resulting Lie algebra as follows

$$\mathfrak{B}[\nu]=\mathbb{R}\,\mathbb{1}\oplus\mathfrak{hs}(\lambda)\,, \tag{B.22}$$

where $\mathbb{1}$ denotes the unit element of the associative algebra $\mathfrak{B}[\nu]$ and the parameter λ is defined by $\lambda:=\frac{1}{2}(\nu+1)$. From our discussion it is then clear that the higher-spin algebra $\mathfrak{hs}(\lambda)$ is isomorphic to the oscillator algebra formed by star commutators of all generators $V_{\alpha(2s-2)}$ in (B.7) with $s\geq 2$.

B.2.3 *Truncation of Higher-Spin Algebra*

In Section 4.1, we have shown that for $\nu = 2N-1$, which corresponds to $\lambda = N$, the higher-spin associative algebra $\mathfrak{B}[2N-1]$ has an non-trivial ideal I_A. The corresponding quotient algebra has the same dimension as $\mathfrak{gl}(N|\mathbb{R})$. In the following, we will show that these two algebras are in fact isomorphic, i.e.

$$\frac{\mathfrak{B}[2N-1]}{I_A} \cong \mathfrak{gl}(N|\mathbb{R})\,. \tag{B.23}$$

From our discussion in Section 4.1, it then follows that the corresponding Lie algebras are also isomorphic, i.e.

$$\frac{\mathfrak{hs}(N)}{I} \cong \mathfrak{sl}(N|\mathbb{R})\,. \tag{B.24}$$

We will prove this for the case $N = 3$ for notational simplicity. But this proof can straightforwardly be extended for arbitrary N.

Theorem. *The associative quotient algebra* $\frac{\mathcal{B}[5]}{I_A}$ *is isomorphic to the associative algebra* $\mathfrak{gl}(3|\mathbb{R})$*, i.e.*

$$\frac{\mathcal{B}[5]}{I_A} \cong \mathfrak{gl}(3|\mathbb{R})\,. \tag{B.25}$$

Proof. Let us consider the following three-dimensional representation of $\mathfrak{sl}(2|\mathbb{R})$:

$$e = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{B.26}$$

It can be easily checked that these matrices obey the expected commutation relations

$$[e,f] = h \qquad [h,e] = 2e \qquad [h,f] = -2f\,. \tag{B.27}$$

These matrices generate the associative algebra $\mathfrak{gl}(3|\mathbb{R})$. This can be seen by first observing that

$$e^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } n > 2. \tag{B.28}$$

We then repeatedly act on e^2 with $adj(f)\bullet = f\bullet - \bullet f$ and obtain

$$\begin{aligned} adj(f)e^2 &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [adj(f)]^2 e^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ [adj(f)]^3 e^2 &= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \qquad [adj(f)]^4 e^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{B.29}$$

Since the quadratic Casimir of the three-dimensional representation is proportional to the unit matrix $\mathbb{1}_{3\times 3}$, we see that e, f and h indeed generate the associative algebra $\mathfrak{gl}(3|\mathbb{R})$. We can turn this associative algebra into a Lie algebra which we then decompose as follows

$$\mathfrak{gl}(3|\mathbb{R}) = \mathbb{R}\,\mathbb{1}_{3\times 3} \oplus \mathfrak{sl}(3|\mathbb{R})\,, \tag{B.30}$$

because e, f, h, e^2 together with the matrices in (B.29) span all traceless 3×3-matrices.

We now consider the higher-spin associative algebra $\mathcal{B}[5]$. The elements of this algebra are words with letters $L_{\alpha\alpha}$, which denote the generators of $\mathfrak{sp}(2|\mathbb{R})$:

$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}\,. \tag{B.31}$$

We define a map φ from the higher-spin algebra $\mathcal{B}[5]$ to $\mathfrak{gl}(3|\mathbb{R})$ by acting on each letter as follows

$$\varphi(L_{\alpha\beta}) = \begin{cases} 2e, & \text{for } \alpha=\beta=0\,, \\ -2f, & \text{for } \alpha=\beta=1\,, \\ -h, & \text{else}\,. \end{cases}$$

This map is well-defined as one can easily check that

$$[\varphi(L_{\alpha\alpha}), \varphi(L_{\alpha\alpha})] = \epsilon_{\alpha\beta}\,\varphi(L_{\alpha\beta})\,, \quad -\frac{1}{2}\varphi(L^{\alpha\beta})\varphi(L_{\alpha\beta}) = 8\,\mathbb{1}_{3\times 3}\,,$$

where we have used that $-\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta} = -\frac{1}{4}(3-2\nu-\nu^2)\mathbb{1} = 8\,\mathbb{1}$ for $\nu = 5$ and with $\mathbb{1}$ denoting the unit element in the higher-spin associative algebra (see Section 3.1.5). This map is obviously a surjective homomorphism.

In the following, we will show that the kernel of this homomorphism is given by the ideal I_A. To this end, we define

$$E = \frac{1}{2}L_{00} \qquad F = -\frac{1}{2}L_{11} \qquad H = -L_{01}\,. \tag{B.32}$$

The algebra decomposes as

$$\mathcal{B}[5] = V_0 \oplus V_1 \oplus V_2 \oplus \ldots\,, \tag{B.33}$$

where the V_n are spanned by

$$E^n\,, \quad adj(F)E^n\,, \quad adj(F)^2E^n\,, \quad \ldots \quad adj(F)^{2n}E^n\,. \tag{B.34}$$

In the following, we will show that the ideal I_A can be written as $I_A = V_3 \oplus V_4 \oplus \ldots$ in this basis. For this we observe that $E^3 \in I_A$ as

$$E^3 = t_{\alpha_1\ldots\alpha_6}\, L^{\alpha_1\alpha_2}L^{\alpha_3\alpha_4}L^{\alpha_5\alpha_6}\,, \tag{B.35}$$

where we define

$$t_{\alpha_1\dots\alpha_6} = \begin{cases} \frac{1}{8} & \text{for } \alpha_1 = \dots = \alpha_6 = 0\,, \\ 0 & \text{else}\,, \end{cases} \tag{B.36}$$

which is obviously completely symmetric and traceless and therefore $E^3 \in I_A$. Since any element of V_n with $n \geq 3$ can be written as a linear combination of products involving E^3, it follows that $V_{n>2} := V_3 \oplus V_4 \oplus \dots \subset I_A$. From

$$dim\, \frac{\mathcal{B}[5]}{I_A} = 9 = dim\,[V_0 \oplus V_1 \oplus V_2] = dim\, \frac{\mathcal{B}[5]}{V_{n>2}}\,, \tag{B.37}$$

we conclude that $V_{n>2} = I_A$. By $\varphi(E^3) = e^3 = 0$ and the fact that E^3 generates the ideal I_A, it follows that $I_A \subset ker\,\varphi$. Since φ is surjective, we deduce from $dim\,[\,V_0 \oplus V_1 \oplus V_2] = dim\,[\mathfrak{gl}(3|\mathbb{R})]$ that $I_A = ker\,\varphi$. This implies that φ induces an isomorphism $\frac{\mathcal{B}[5]}{I_A} \cong \mathfrak{gl}(3|\mathbb{R})$. □

This proof can easily generalized for general $N \in \mathbb{N}$ by considering a N-dimensional representation of $\mathfrak{sl}(2|\mathbb{R})$.

B.3 VANISHING OF R

In this section, we will give a proof for equation (3.122). In Fourier space R is given by (A.27) with the kernel

$$K_{\alpha\alpha} = \int_0^1 \mathrm{d}t\, (t^2-1)(y_\alpha y_\alpha + \xi_\alpha \xi_\alpha)\cos(ty\xi)\,. \tag{B.38}$$

Note that the cosine is due to the fact that we are considering bosonic fields. We can now use the Fourier representation of $\tilde{D}$ which is proportional to

$$E^{\alpha\beta}\,\epsilon^{\alpha\beta}\,(\mathcal{O}^y_{\alpha\alpha} - \mathcal{O}^\xi_{\alpha\alpha})K_{\beta\beta}\,, \tag{B.39}$$

where we have defined

$$\mathcal{O}^y_{\alpha\alpha} = \frac{i}{2}\,\phi(y_\alpha y_\alpha - \partial^y_\alpha \partial^y_\alpha) \tag{B.40}$$

and $\mathcal{O}^\xi_{\alpha\alpha}$ is analogously defined. This can be seen by comparing with (A.29d). Since the kernel $K_{\alpha\alpha}$ is symmetric with respect to exchange of ξ and y, we obtain

$$\int_0^1 (t^2-1)E^{\alpha\beta}\epsilon^{\alpha\beta}\left(\mathcal{O}^y_{\alpha\alpha}(y_\beta y_\beta + \xi_\beta\xi_\beta)\cos(ty\xi) - y \leftrightarrow \xi\right)\,,$$

by applying the Fourier representation of $\tilde{D}$ on R. There are two summands in $\mathcal{O}^y_{\alpha\alpha}$. We will consider them separately and drop overall factors. The first summand leads to

$$\begin{aligned}&\int_0^1 (t^2-1)E^{\alpha\beta}\epsilon^{\alpha\beta}\left(y_\alpha y_\alpha(y_\beta y_\beta + \xi_\beta\xi_\beta)\cos(ty\xi) - y \leftrightarrow \xi\right) \\ &= -\,2\int_0^1 (t^2-1)\,E^{\alpha\alpha}\,y_\alpha\xi_\alpha\,\frac{\mathrm{d}}{\mathrm{d}t}\sin(ty\xi)\,,\end{aligned}$$

After some algebra the second summand is given by

$$\begin{aligned}&\int_0^1 (1-t^2)\,E^{\alpha\beta}\epsilon^{\alpha\beta}\,\left(\partial^y_\alpha\partial^y_\alpha(y_\beta y_\beta+\xi_\beta\xi_\beta)\cos(ty\xi)-y\leftrightarrow\xi\right)\\=&\int_0^1(1-t^2)\,E^{\alpha\alpha}\,\xi_\alpha y_\alpha\,\frac{1}{t}(2+t\frac{\mathrm{d}}{\mathrm{d}t})t^2\,\sin(ty\xi)\,.\end{aligned}$$

Adding the two summands we obtain up to overall factors

$$\begin{aligned}&E^{\alpha\alpha}\,\xi_\alpha y_\alpha\int_0^1\left\{(t^2-1)^2\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\left[(t^2-1)^2\right]\right\}\sin(ty\xi)\\=&E^{\alpha\alpha}\,\xi_\alpha y_\alpha\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\left\{(t^2-1)^2\sin(ty\xi)\right\}\\=&0\,,\end{aligned}$$

which shows that (3.122) holds.

B.4 BREAKING OF LORENTZ SYMMETRY

In this section, we illustrate how the term (5.5) which breaks manifest local Lorentz symmetry arises by expanding (5.2b) to second order

$$D_\Omega\boldsymbol{\omega}^{(2)}=-D_\Omega\left(z^\alpha\Gamma_0\langle D_\Omega\mathcal{A}^{(2)}_\alpha-[\mathcal{W}^{(1)},\mathcal{A}^{(1)}_\alpha]_\star\rangle\right)+\mathcal{W}^{(1)}\wedge\star\mathcal{W}^{(1)}\,. \tag{B.41}$$

Let us only focus on terms which will lead to contributions involving the background spin-connection $\bar{\omega}$ or the exterior derivative d

$$\begin{aligned}\nabla\boldsymbol{\omega}^{(2)}=&-\nabla\left(z^\alpha\Gamma_0\langle\nabla\mathcal{A}^{(2)}_\alpha\rangle\right)-\nabla\left(z^\alpha\Gamma_0\langle[z^\beta\Gamma_0\langle\nabla\mathcal{A}^{(1)}_\beta\rangle,\mathcal{A}^{(1)}_\alpha]_\star\rangle\right)\\&+z^\alpha\Gamma_0\langle\nabla\mathcal{A}^{(1)}_\alpha\rangle\wedge\star\,z^\beta\Gamma_0\langle\nabla\mathcal{A}^{(1)}_\beta\rangle+\dots\,,\end{aligned}$$

where we have used that the only term in $\mathcal{W}^{(1)}$, which contains the background spin-connection or the exterior derivative, is $z^\beta\Gamma_0\langle\nabla\mathcal{A}^{(1)}_\beta\rangle$. The first summand of the right hand side of the equation above will vanish for the same reason which ensured that there was no such term at linear order. We recall that the Lorentz covariant derivative is defined as $\nabla\bullet=\mathrm{d}\bullet-[\bar{\omega},\bullet]_\star$. All terms involving the exterior differential d on the right hand side will not contribute. This can be seen by using $\mathrm{d}(z_\alpha\bullet)=z_\alpha\mathrm{d}\bullet$ which together with $z^\alpha A^{(1)}_\alpha=0$ implies that

$$z^\alpha\Gamma_0\langle\mathrm{d}\mathcal{A}^{(1)}_\alpha\rangle=\Gamma_0\langle\tfrac{1}{t}\mathrm{d}(z^\alpha\mathcal{A}^{(1)}_\alpha)\rangle=0\,. \tag{B.42}$$

Similarly, one immediately concludes that for $z=0$ all other terms involving the external derivative d drop out because of

$$\mathrm{d}(z_\alpha\dots)|_{z=0}=z_\alpha\mathrm{d}(\dots)|_{z=0}=0\,. \tag{B.43}$$

We therefore arrive at

$$\begin{aligned}\nabla\boldsymbol{\omega}^{(2)}=&-[\bar{\omega},z^\alpha\Gamma_0\langle[z^\beta\Gamma_0\langle[\bar{\omega},\mathcal{A}^{(1)}_\beta]_\star\rangle,\mathcal{A}^{(1)}_\alpha]_\star\rangle]_\star\\&+z^\alpha\Gamma_0\langle\left[\bar{\omega},\mathcal{A}^{(1)}_\alpha\right]_\star\rangle\wedge\star\,z^\beta\Gamma_0\langle\left[\bar{\omega},\mathcal{A}^{(1)}_\beta\right]_\star\rangle+\dots\,,\end{aligned} \tag{B.44}$$

where we have again not made any terms explicit which involve background vielbeins. By an explicit calculation, it can be shown that these contributions lead to (5.5). Using the conventions of Appendix B.1.1, the explicit form of $T_{\alpha\alpha}(\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(1)})$ is

$$T_{\alpha\alpha} = i \int \mathrm{d}^2\xi\, \mathrm{d}^2\eta\, \mathrm{d}t\, \mathrm{d}q \Big((\eta+\xi)_\alpha (\eta+y)_\alpha\, t^2 q^2\, Q \\ - (\xi-\eta)_\alpha (\xi-\eta)_\alpha\, t^2\, P \Big) \hat{C}^{(1)}(\xi) \hat{C}^{(1)}(\eta)\,.$$

but is not of critical importance apart from the fact that it is bilinear in $\hat{C}^{(1)}$ and non-vanishing.

C

σ_--COHOMOLOGY

In this appendix, we will discuss a cohomological analysis [90, 91] which allows one to elegantly identify the field content and dynamical equations encoded by fields and equations of the unfolded formalism.

C.1 DEFINITIONS

We consider general unfolded equations of the form

$$\mathcal{D}C = 0\,, \tag{C.1}$$

where the field C denotes an arbitrary p-form which splits with respect to a certain grading

$$C = \sum_{n=0}^{\infty} C_n\,. \tag{C.2}$$

All fields arising in Vasiliev theory are of this form. We denote by $\mathcal{D}$ a general nilpotent differential which decomposes as

$$\mathcal{D}\bullet = \nabla\bullet + \sigma_-\bullet + \sigma_+\bullet\,, \tag{C.3}$$

where ∇ denotes the Lorentz covariant derivative and σ_+ and σ_- raises and lowers the grade by one respectively.[1] Nilpotency of the differential implies

$$\mathcal{D}^2 = 0 \quad \Leftrightarrow \quad \sigma_\pm^2 = 0\,, \quad \{\nabla, \sigma_\pm\} = 0\,, \quad \nabla^2 = -\{\sigma_+, \sigma_-\}\,. \tag{C.4}$$

By nilpotency of $\mathcal{D}$, it also follows for $p \geq 1$ that the unfolded equation of motion (C.1) is gauge invariant under

$$\delta C = \mathcal{D}\xi\,, \tag{C.5}$$

where ξ is of form-degree $p-1$. We also define the operator $\sigma_-^{\#}$ by

$$\sigma_-(\sigma_-^{\#}(J)) = J \tag{C.6}$$

for all σ_--exact J. Note that $\sigma_-^{\#}$ is only defined up to σ_--closed terms S

$$\sigma_-(\sigma_-^{\#}(J) + S) = J\,. \tag{C.7}$$

We denote by H_p^n the set of all equivalence classes of σ_--closed p-forms of grade n identified by σ_--exact terms of the same form-degree and grade. This set is usually referred to as the *σ_--cohomology* of form-degree p and grade n and forms a vector space. Furthermore, we denote the direct sum of these vector spaces by $H_p := \oplus_n H_p^n$ and refer to it as the *σ_--cohomology* of form-degree p.

1 One can straightforwardly generalize the argument for operators $\sigma_\pm$ which change the grade by more than one unit. But we will not do so for simplicity.

C.2 THEOREMS

These definitions are interesting because

1. The basis elements of H^n_p are in a one-to-one correspondence with the components of the field C_n that *cannot* be expressed in terms of C_i with $i < n$ and are not Stuckelberg fields[2]. Such components are usually referred to as dynamical fields.

2. The basis elements of H^n_{p+1} are in a one-to-one correspondence with the components of $(\mathcal{D}C)_n = 0$ that are *not* a consequence of $(\mathcal{D}C)_i = 0$ for $i < n$. These components are commonly referred to as dynamical equations.

We will prove these statements in the following.

PROOF OF 1: From the unfolded equation of motion (C.1) at grade $n-1$, it follows that

$$(\mathcal{D}C)_{n-1} = 0 \qquad \Leftrightarrow \qquad \nabla C_{n-1} + \sigma_- C_n + \sigma_+ C_{n-2} = 0\,. \tag{C.8}$$

Therefore, we obtain

$$C_n = -\sigma_-^{\#}\left(\nabla C_{n-1} + \sigma_+ C_{n-2}\right) + S\,, \tag{C.9}$$

where S is σ_--closed. But for $p \geq 1$, the field C_n has the following gauge transformation

$$\delta C_n = (\mathcal{D}\xi)_n = \cdots + \sigma_- \xi_{n+1} + \ldots\,, \tag{C.10}$$

which allow us to shift S by σ_--exact terms. So, up to gauge transformations, C_n is uniquely determined by the fields C_i with $i < n$ if the cohomology H^n_p is trivial. If this cohomology is non-trivial and k-dimensional, we arrive at

$$C_n = -\sigma_-^{\#}\left(\nabla C_{n-1} + \sigma_+ C_{n-2}\right) + \sum_{i=1}^{k} c_i \mathcal{C}^i_n\,, \tag{C.11}$$

where the c_i are free coefficients and the $\mathcal{C}^i_n$ form a basis of H^n_p. Therefore in this case, the components $c_i \mathcal{C}^i_n$ cannot be expressed in terms of the C_i with $i < n$ nor be gauged away by the algebraic gauge transformation $\sigma_- \xi_{n+1}$.

PROOF OF 2: Nilpotency of $\mathcal{D}$ implies that

$$\mathcal{D}^2 C = 0 \quad \Leftrightarrow \quad \sigma_-(\mathcal{D}C)_n = -\nabla(\mathcal{D}C)_{n-1} - \sigma_+(\mathcal{D}C)_{n-2}\,. \tag{C.12}$$

This implies

$$(\mathcal{D}C)_n = -\sigma_-^{\#}\left(\nabla(\mathcal{D}C)_{n-1} + \sigma_+(\mathcal{D}C)_{n-2}\right) + S\,, \tag{C.13}$$

2 By Stuckelberg fields we mean fields that can be expressed as purely algebraic gauge transformations.

with S σ_--closed. The σ_--exact part of S can be removed by field redefinitions $C_{n+1} \to C_{n+1} + \delta C_{n+1}$ as $(\mathcal{D}C)_n = \dots + \sigma_- C_{n+1} \dots$ implies that

$$(\mathcal{D}C)_n \qquad \longrightarrow \qquad (\mathcal{D}C)_n + \sigma_- \delta C_{n+1}\,. \tag{C.14}$$

If H^n_{p+1} is trivial, then $(\mathcal{D}C)_n$ is uniquely determined in terms of $(\mathcal{D}C)_i$ with $i < n$ (up to field redefinitions). If H^n_{p+1} is non-trivial and k-dimensional, we obtain

$$(\mathcal{D}C)_n = -\sigma_-^{\#}\left(\nabla(\mathcal{D}C)_{n-1} - \sigma_+(\mathcal{D}C)_{n-2}\right) + \sum_{i=1}^{k} c_i \mathcal{E}^i_n(C_n, C_{n-1}, \dots)\,,$$

where c_i are free coefficients and $\mathcal{E}^i_n(C_n, C_{n-1}, \dots)$ form a basis of H^n_{p+1}. The equation $(\mathcal{D}C)_n = 0$ therefore imposes k dynamical equation of motions

$$\mathcal{E}^i(C_n, C_{n-1}, \dots) = 0 \qquad\qquad i = 1, \dots, k\,. \tag{C.15}$$

* * *

These theorems provide us with a powerful tool to obtain the full set of dynamical equations of motion and fields encoded by the unfolded equations. We will illustrate this for various examples in the following.

C.3 EXAMPLES

C.3.1 *Three-Dimensional Vasiliev Theory*

C.3.1.1 *Zero-Form Sector*

The equations of motion of the physical zero-form sector are given by (3.57) which can be rewritten as[3]

$$-\frac{1}{2}\bar{e}^{\alpha\alpha}\nabla_{\alpha\alpha}\hat{C}(y) = \frac{\phi}{2i\,l}\,\bar{e}^{\alpha\alpha}(y_\alpha y_\alpha - \partial_\alpha\partial_\alpha)\hat{C}(y)\,. \tag{C.16}$$

As a result, we see that the equation of motion is indeed of the general form (C.1) and that the $\hat{C}$ field splits as $\hat{C} = \sum_{n=0}^{\infty} \hat{C}_n$ with $N\hat{C}_n = n\hat{C}_n$ where $N = y^\alpha \partial_\alpha$ is the y-number operator. We then identify

$$\sigma_- \bullet = \bar{e}^{\alpha\alpha}\partial_\alpha\partial_\alpha \bullet\,, \tag{C.17}$$

where we have for convenience chosen a slightly different normalization for σ_- as compared to (C.3) and σ_- now lowers the grading by two.

3 Here we have used (3.64) to obtain $\mathrm{d}x^n \nabla_n = -\frac{1}{2}\mathrm{d}x^n \bar{e}^{\alpha\alpha}_n \bar{e}^m_{\alpha\alpha}\nabla_m = -\frac{1}{2}\bar{e}^{\alpha\alpha}\nabla_{\alpha\alpha}$.

ZERO-FORM COHOMOLOGY: From the definition of σ_- it is clear that

$$c_\alpha y^\alpha \qquad\qquad \text{and} \qquad\qquad c \tag{C.18}$$

parameterize all σ_--closed zero-forms. Here c_α and c denote constants with respect to y. Since there are no σ_--exact zero-forms, they also parameterize the cohomology H_0. As we will show, they correspond to a Weyl fermion $\hat{C}_\alpha$ and scalar field $\Phi = \hat{C}(y=0)$ respectively.

ONE-FORM COHOMOLOGY: The most general one-form is given by

$$f(y) = f_n(y)\mathrm{d}x^n = \bar{e}^{\alpha\alpha}\Big(y_\alpha y_\alpha\, f^{yy}(y) + y_\alpha \partial_\alpha\, f^{y\partial}(y) + \partial_\alpha\partial_\alpha\, f^{\partial\partial}(y)\Big)\,. \tag{C.19}$$

This expansion determines all summands in the decomposition above uniquely[4]. Using the definition of σ_- one determines

$$\sigma_- f \sim E^{\alpha\alpha}\Big\{\partial_\alpha\partial_\alpha(N+1)f^{y\partial} + y_\alpha\partial_\alpha(N+3)f^{yy}\Big\}\,. \tag{C.20}$$

Since the summands in this decomposition are also unique, we conclude that σ_--closed one-forms are parameterized by[5]

$$f^{y\partial} = c_\alpha y^\alpha \qquad\qquad f^{yy} = c \qquad\qquad f^{\partial\partial} = g(y), \tag{C.21}$$

where c_α and c again denote y-independent constants and $g(y)$ is an arbitrary function in y. From the definition of σ_-, it is also clear that $f^{\partial\partial}$ parameterizes exact one-forms. Therefore, the first two elements in (C.21) parameterize a basis of the cohomology H_1 and, as we will show now, are indeed in one-to-one correspondence with a Weyl and Klein–Gordon equation respectively.

Let us start with the Weyl equation. Note that the right hand side of the unfolded equation (C.16) is already of the form (C.19). To bring also its left hand side in the same form, we use the following identity

$$\begin{aligned}\bar{e}^{\alpha\alpha}g_{\alpha\alpha} = \bar{e}^{\alpha\alpha}\Big(&\partial_\alpha\partial_\alpha\tfrac{1}{N(N-1)}y^\beta y^\beta g_{\beta\beta}\\ &+ y_\alpha\partial_\alpha\tfrac{2}{N(N+2)}y^\beta\partial_\gamma g_\beta{}^\gamma\\ &+ y_\alpha y_\alpha\tfrac{1}{(N+2)(N+3)}\partial_\beta\partial_\beta g^{\beta\beta}\Big)\,.\end{aligned} \tag{C.22}$$

4 For any

$$y_\alpha y_\alpha\, f^{yy}(y) + y_\alpha\partial_\alpha\, f^{y\partial}(y) + \partial_\alpha\partial_\alpha\, f^{\partial\partial}(y) = 0$$

we can contract with y^α to obtain

$$\partial_\alpha(N-1)f^{\partial\partial} = 0 \qquad\qquad \Rightarrow \qquad\qquad \partial_\alpha\partial_\alpha f^{\partial\partial} = 0\,.$$

Similarly, we can contract with ∂^α to obtain

$$y_\alpha f^{yy} = 0 \qquad\qquad \Rightarrow \qquad\qquad y_\alpha y_\alpha f^{yy} = 0\,.$$

This implies that all summands have to vanish independently.

5 The case $f^{y\partial} = c$ for an arbitrary constant c is excluded as $\partial_\alpha\partial_\alpha c = 0$.

Using this identity, we can project the unfolded equation of motion (C.16) on its $f^{y\partial}$ component linear in y which gives

$$\bar{e}^{\alpha\alpha} y_\alpha \partial_\alpha \left[\tfrac{2}{N(N+2)} y^\beta \nabla_\beta{}^\gamma \partial_\gamma \hat{C}(y)\right]_{\text{linear in } y} = 0\,, \tag{C.23}$$

which is indeed equivalent to the Weyl equation

$$\nabla_\beta{}^\gamma \hat{C}_\gamma = 0\,. \tag{C.24}$$

Similarly, the Klein–Gordon equation can be derived as follows: the projection on the $f^{yy}|_{y=0}$ component gives

$$-\frac{1}{2}\bar{e}^{\alpha\alpha} y_\alpha y_\alpha \left[\tfrac{1}{(N+2)(N+3)} \nabla^{\beta\beta} \partial_\beta \partial_\beta \hat{C}(y)\right]_{y=0} = \tfrac{\phi}{2il}\, \bar{e}^{\alpha\alpha} y_\alpha y_\alpha \Phi\,, \tag{C.25}$$

where we have used the notation $\Phi = \hat{C}(0)$. This is equivalent to

$$-\frac{1}{12}\nabla^{\alpha\alpha}\hat{C}_{\beta\beta} = \frac{\phi}{2il}\Phi\,. \tag{C.26}$$

Together with $\hat{C}_{\beta\beta} = 4i\,\phi\, l\, \nabla_{\beta\beta}\Phi$ (see (3.63)) and $\Box = -\frac{1}{2}\nabla_{\alpha\alpha}\nabla^{\alpha\alpha}$, we arrive at

$$\Box\Phi = -\frac{3}{4l^2}\Phi\,, \tag{C.27}$$

which is the Klein–Gordon equation (3.67) for a complex scalar field. Therefore, the unfolded equations for the physical zero-form are equivalent to a Klein–Gordon equation and Weyl equation for a scalar and Weyl fermion field respectively. We considered only the bosonic theory in the main text for which $\hat{C}(y)$ is an even function of y. In this case the unfolded equations (C.16) are equivalent to the Klein–Gordon equation (C.27).

C.3.2 *Four-Dimensional Vasiliev Theory*

C.3.2.1 *Zero-Form Sector*

The unfolded equations (6.28) for the zero-form $C(Y|x)$ of four dimensional Vasiliev theory can be rewritten as

$$-\frac{1}{2}\bar{e}^{\alpha\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}C(y,\bar{y}) = -\frac{i}{l}\,\bar{e}^{\alpha\dot{\alpha}}\left(y_\alpha\bar{y}_{\dot{\alpha}} - \partial_\alpha\partial_{\dot{\alpha}}\right)C(y,\bar{y})\,, \tag{C.28}$$

the factor of $-\frac{1}{2}$ on the left hand side of the equation arises for the same reason as in the three-dimensional case. This equation only relates components with the same $2s = n - \bar{n}$ in the expansion (6.29) which we repeat here for convenience

$$C(Y|x) = \sum_{\substack{n,\bar{n}=0\\ n+\bar{n}\in 2\mathbb{N}}}^{\infty} \frac{1}{n!\bar{n}!} C_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_{\bar{n}}}(x)\, y^{\alpha_1}\ldots y^{\alpha_n}\,\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_{\bar{n}}}\,. \tag{C.29}$$

Note that we are only considering the bosonic case for simplicity. We will consider fixed spin s in the following. In this case, the total number of y (or $\bar{y}$) oscillators induces a natural grading. We then define[6]

$$\sigma_- \bullet = \bar{e}^{\alpha\dot{\alpha}} \partial_\alpha \partial_{\dot{\alpha}} \bullet \ . \tag{C.30}$$

ZERO-FORM COHOMOLOGY: From the definition of σ_-, it is clear that σ_- closed zero forms are given by

$$\sigma_- C(y, \bar{y}) = 0 \quad \Rightarrow \quad C(y, \bar{y}) = f(y) \quad , \quad C(y, \bar{y}) = g(\bar{y}) \tag{C.31}$$

where $f(y)$ and $g(\bar{y})$ are arbitrary functions. Since we are considering zero-forms, the set of closed zero-forms is isomorphic to the cohomology H_0. As we will see, upon expanding in y and $\bar{y}$ the expansion coefficients $C^{\alpha(2s)}$, $C^{\dot{\alpha}(2s)}$ and $\Phi = C(0,0)$ describe spin-s Weyl tensors and a scalar field respectively.

ONE-FORM COHOMOLOGY: The most general one-form is given by

$$f_n(y) \mathrm{d}x^n = \bar{e}^{\alpha\dot{\alpha}} \left(y_\alpha \bar{y}_{\dot{\alpha}} f^{y\bar{y}} + \bar{y}_{\dot{\alpha}} \partial_\alpha f^{\bar{y}\partial} + y_\alpha \partial_{\dot{\alpha}} f^{y\bar{\partial}} + \partial_\alpha \partial_{\dot{\alpha}} f^{\partial\bar{\partial}} \right) \ . \tag{C.32}$$

This expansion determines all summands uniquely.[7] Using the definition of σ_- and the identity (C.22), it is easy to derive

$$\begin{aligned} \sigma_- f(y) \sim E^{\alpha\alpha} & \left(y_\alpha \partial_\alpha (\bar{N}+2) f^{y\bar{y}} + \partial_\alpha \partial_\alpha (\bar{N}+2) f^{\bar{y}\partial} \right) \\ & + \bar{E}^{\dot{\alpha}\dot{\alpha}} \left(\bar{y}_{\dot{\alpha}} \partial_{\dot{\alpha}} (N+2) f^{y\bar{y}} + \partial_{\dot{\alpha}} \partial_{\dot{\alpha}} (N+2) f^{y\bar{\partial}} \right) \ , \end{aligned} \tag{C.37}$$

where $N = y^\alpha \partial_\alpha$ and $\bar{N} = \bar{y}^\alpha \partial_{\dot{\alpha}}$. It can again be shown that the summands in the above decomposition are unique. Therefore for σ_--closed one-forms we obtain (among others) the following condition

$$\partial_\alpha \partial_\alpha (\bar{N}+2) f^{\bar{y}\partial} = 0 \quad \Rightarrow \quad f^{\bar{y}\partial} \sim c_\alpha y^\alpha f(\bar{y}) + c\, g(\bar{y}) \, , \tag{C.38}$$

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6 For convenience we have again chosen a slightly different normalization for σ_- as compared to (C.3).

7 This can be seen as follows: we start from

$$y_\alpha \bar{y}_{\dot{\alpha}} f^{y\bar{y}} + \bar{y}_{\dot{\alpha}} \partial_\alpha f^{\bar{y}\partial} + y_\alpha \partial_{\dot{\alpha}} f^{y\bar{\partial}} + \partial_\alpha \partial_{\dot{\alpha}} f^{\partial\bar{\partial}} = 0 \, . \tag{C.33}$$

By contracting this equation with y^α we obtain

$$\bar{y}_{\dot{\alpha}} N f^{\bar{y}\partial} + \partial_{\dot{\alpha}} N f^{\partial\bar{\partial}} = 0 \, . \tag{C.34}$$

But, by the compatibility condition $(\bar{N}+2) N f^{\bar{y}\partial} = 0$ of this partial differential equation, it follows that

$$N f^{\bar{y}\partial} = 0 \qquad \Rightarrow \qquad \bar{y}_{\dot{\alpha}} \partial_\alpha f^{\bar{y}\partial} = 0 \, , \tag{C.35}$$

$$\partial_{\dot{\alpha}} N f^{\partial\bar{\partial}} = 0 \qquad \Rightarrow \qquad \partial_\alpha \partial_{\dot{\alpha}} f^{\partial\bar{\partial}} = 0 \, . \tag{C.36}$$

By contracting with $\bar{y}^{\dot{\alpha}}$, analogous statements follow for $f^{y\bar{y}}$ and $f^{y\bar{\partial}}$.

where c and c_α are constants and $f(\bar{y}), g(\bar{y})$ arbitrary functions. The term proportional to $g(\bar{y})$ does not contribute to the corresponding one-form as $\bar{e}^{\alpha\dot{\alpha}}\bar{y}_{\dot{\alpha}}\partial_\alpha(c\, g(\bar{y})) = 0$. By completely analogous reasoning for the other conditions, it follows that closed one-forms are parameterized by

$$f^{\partial\bar{\partial}} = g(y,\bar{y})\,, \quad f^{y\bar{y}} = c\,, \quad f^{\bar{y}\partial} = c_\alpha y^\alpha f(y)\,, \quad f^{y\bar{\partial}} = \bar{c}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}\bar{f}(\bar{y})\,. \tag{C.39}$$

But from the definition of σ_-, it is clear that $f^{\partial\bar{\partial}}$ parameterizes exact one-forms. Therefore, only the last three elements (i.e. $f^{y\bar{y}} = c$, $f^{\bar{y}\partial} = c_\alpha y^\alpha f(y)$ and $f^{y\bar{\partial}} = \bar{c}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}\bar{f}(\bar{y})$) parameterize a basis for the cohomology H_1.

Let us first show that the $f^{y\bar{y}}|_{y=\bar{y}=0}$ component is associated with a Klein–Gordon equation. Using twice the identity

$$f_\alpha(y) = \partial_\alpha\left(\tfrac{1}{N}y^\beta f_\beta\right) + y_\alpha\left(\tfrac{1}{N+2}\partial_\beta f^\beta\right)\,, \tag{C.40}$$

we can project the unfolded equation (C.28) on the $f^{y\bar{y}}|_{y=\bar{y}=0}$ component which gives

$$\tfrac{-1}{2}\bar{e}^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}\left[\tfrac{1}{(N+2)(\bar{N}+2)}\nabla^{\beta\dot{\beta}}\partial_\beta\partial_{\dot{\beta}}C(y,\bar{y})\right]_{\bar{y}=y=0} = \tfrac{-i}{l}\,\bar{e}^{\alpha\dot{\alpha}}\,y_\alpha\bar{y}_{\dot{\alpha}}\Phi\,,$$

which is equivalent to

$$-\frac{1}{8}\nabla^{\alpha\dot{\alpha}}C_{\alpha\dot{\alpha}} = -\frac{i}{l}\Phi\,. \tag{C.41}$$

Using $C_{\alpha\dot{\alpha}} = 2il\,\nabla_{\alpha\dot{\alpha}}\Phi$ (see (6.35)) and $-\frac{1}{2}\nabla^{\alpha\dot{\alpha}}\nabla^{\alpha\dot{\alpha}} = \Box$, we arrive at

$$\Box\Phi = -\frac{2}{l^2}\Phi\,. \tag{C.42}$$

The other two elements parameterizing the cohomology correspond to the equations of the Weyl tensors. Let us illustrate this by projecting the unfolded equation (C.28) on its $f^{\bar{y}\partial} = c_\alpha y^\alpha f(y)$ component. To this end, we apply the identity (C.40) twice again. This gives

$$-\frac{1}{2}\bar{e}^{\alpha\dot{\alpha}}\bar{y}_{\dot{\alpha}}\partial_\alpha\left[\tfrac{1}{(N+2)(\bar{N}+2)}y^\beta\partial_{\dot{\beta}}\nabla_\beta{}^{\dot{\beta}}C(y,\bar{y})\right]_{(N,\bar{N})=(1,n)} = 0\,, \tag{C.43}$$

where $[f(y,\bar{y})]_{(N,\bar{N})=(1,n)}$ denotes the term in $f(y,\bar{y})$ which is linear in y and of order n in $\bar{y}$. This relation is obviously equivalent to

$$\nabla_{\alpha\dot{\beta}}C^{\dot{\beta}}{}_{\dot{\alpha}(n)} = 0\,. \tag{C.44}$$

We have therefore shown that the unfolded equations for the zero-form (C.28) are equivalent to a Klein–Gordon equation and (generalized) Bianchi identities.

C.3.2.2 *One-Form Sector*

While the analysis of the σ_--cohomology for the one-form sector follows very similar lines as for the zero-form sector and is completely straightforward, the necessary calculations are a bit involved and we will therefore only present the results. The one-form field is given by

$$\omega(y,\bar{y}|x) = \omega(y,\bar{y}|x)_n \mathrm{d}x^n \tag{C.45}$$

and obeys the following unfolded equations of motion[8]

$$D\omega = -\frac{1}{2}\, e^{i\theta} E^{\alpha\alpha}\partial_\alpha\partial_\alpha C(y,0) - \frac{1}{2}\, e^{-i\theta} E^{\dot\alpha\dot\alpha}\partial_{\dot\alpha}\partial_{\dot\alpha} C(0,\bar{y})\,, \tag{C.46}$$

which is gauge invariant under

$$\delta\omega(y,\bar{y}) = D\xi(y,\bar{y})\,, \tag{C.47}$$

for a zero-form $\xi(y,\bar{y})$. In the following, it is convenient to restrict ourselves to components of the one-form ω which encode the spin-s degrees of freedom, i.e. to those ω which obey $\frac{1}{2}(N+\bar{N})\omega = (s-1)\,\omega$. Fixing the spin the one-form ω then splits as

$$\omega = \sum_{k=-s+1}^{s-1} \omega_k \qquad \text{with} \qquad \frac{1}{2}(N-\bar{N})\omega_k = k\,\omega_k\,. \tag{C.48}$$

For example for $s=2$, these components correspond to $\omega_1 \sim \omega_{\alpha\alpha} y^\alpha y^\alpha$, $\omega_{-1} \sim \omega_{\dot\alpha\dot\alpha}\bar{y}^{\dot\alpha}\bar{y}^{\dot\alpha}$ and $\omega_0 \sim \bar{e}_{\alpha\dot\alpha} y^\alpha \bar{y}^{\dot\alpha}$. The first two encode the spin-connection $\omega^{a,b}$ and the last the vielbein e^a. The adjoint covariant derivative is given by

$$D = \nabla + Q_+ + Q_-\,, \tag{C.49}$$

where the $Q_\pm$ were defined in (6.106). We now define the σ_--operator acting on Lorentz tensors $\omega^{a(s-1),b(k+1)}$ by

$$\sigma_-(\omega)^{a(s-1),b(k)} \sim \begin{cases} \bar{e}_c \wedge \omega^{a(s-1),b(k)c} & 0 \le t < s-1 \\ 0 & t = s-1 \end{cases}. \tag{C.50}$$

This definition agrees with (2.46) of Chapter 2. In the spin-s one-form $\omega(Y)$ the components

$$\omega_{+k} \sim \omega_{\alpha(s-1+k)\dot\alpha(s-1-k)}\, y^{\alpha(s-1+k)}\bar{y}^{\dot\alpha(s-1-k)} \tag{C.51}$$

$$\omega_{-k} \sim \omega_{\alpha(s-1-k)\dot\alpha(s-1+k)}\, y^{\alpha(s-1-k)}\bar{y}^{\dot\alpha(s-1+k)} \tag{C.52}$$

correspond to $\omega^{a(s-1),b(k)}$ in accordance with the map from Lorentz to spinorial indices (see Section 6.1). Therefore in spinorial language the operator σ_- corresponds to

$$\sigma_-(\omega_n,\omega_{-n}) = \begin{cases} Q_-\omega_n + Q_+\omega_{-n} & n > 0 \\ 0 & n = 0 \end{cases}. \tag{C.53}$$

8 For readers who have not yet read through the discussion of the non-linear equations the phase factor $e^{i\theta}$ in this relation might be surprising. We refer to Section 6.8 for an explanation.

ONE-FORM COHOMOLOGY: We use the decomposition

$$f_k = \bar{e}^{\alpha\dot{\alpha}} \left(y_\alpha \bar{y}_{\dot{\alpha}} f_k^{y\bar{y}} + \bar{y}_{\dot{\alpha}} \partial_\alpha f_k^{\bar{y}\partial} + y_\alpha \partial_{\dot{\alpha}} f_k^{y\bar{\partial}} + \partial_\alpha \partial_{\dot{\alpha}} f_k^{\partial\bar{\partial}} \right) , \tag{C.54}$$

with $\frac{1}{2}(N - \bar{N}) f_k = k\, f_k$. It can be shown that the cohomology H_1 is parameterized by

$$f_0^{\partial\bar{\partial}} = g(y, \bar{y}) \qquad\qquad f_0^{y\bar{y}} = h(y, \bar{y}) \,, \tag{C.55}$$

where g and h are arbitrary functions obeying $(N - \bar{N})g = (N - \bar{N})h = 0$. The dynamical fields therefore are contained in the one-form as follows[9]

$$\omega_0(y, \bar{y}) = \bar{e}^{\alpha\dot{\alpha}} \left(\partial_\alpha \partial_{\dot{\alpha}}\, \phi(y, \bar{y}) + y_\alpha \bar{y}_{\dot{\alpha}}\, \phi'(y, \bar{y}) + \dots \right) , \tag{C.56}$$

and ϕ and ϕ' correspond to the traceless and trace components of the Fronsdal field respectively. We will explain this interpretation in more detail in the next section.

TWO-FORM COHOMOLOGY: We decompose a general two-form by

$$\begin{aligned} J_k = \; & E^{\alpha\alpha} \partial_\alpha \partial_\alpha J_k^{\partial\partial} + E^{\alpha\alpha} y_\alpha \partial_\alpha J_k^{y\partial} + E^{\alpha\alpha} y_\alpha y_\alpha J_k^{yy} \\ & + E^{\dot{\alpha}\dot{\alpha}} \partial_{\dot{\alpha}} \partial_{\dot{\alpha}} \bar{J}_k^{\partial\partial} + E^{\dot{\alpha}\dot{\alpha}} \bar{y}_{\dot{\alpha}} \partial_{\dot{\alpha}} \bar{J}_k^{y\partial} + E^{\dot{\alpha}\dot{\alpha}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\alpha}} \bar{J}_k^{yy} \,. \end{aligned} \tag{C.57}$$

The cohomology H_2 can then be parameterized by

$$J_{s-1}^{\partial\partial} = \mathcal{C}(y) \qquad \leftrightarrow \qquad C_{\alpha(2s)} \tag{C.58a}$$

$$J_1^{\partial\partial} = \mathcal{F}(y, \bar{y}) \qquad \leftrightarrow \qquad F_{\alpha(s)\dot{\alpha}(s)} \tag{C.58b}$$

$$J_1^{\bar{y}\bar{y}} = \mathcal{F}'(y, \bar{y}) \qquad \leftrightarrow \qquad F'_{\alpha(s-2)\dot{\alpha}(s-2)} \tag{C.58c}$$

$$J_{-1}^{yy} = \mathcal{F}'(y, \bar{y}) \qquad \leftrightarrow \qquad F'_{\alpha(s-2)\dot{\alpha}(s-2)} \tag{C.58d}$$

$$J_{-1}^{\bar{\partial}\bar{\partial}} = \mathcal{F}(y, \bar{y}) \qquad \leftrightarrow \qquad F_{\alpha(s)\dot{\alpha}(s)} \tag{C.58e}$$

$$J_{1-s}^{\bar{\partial}\bar{\partial}} = \bar{\mathcal{C}}(\bar{y}) \qquad \leftrightarrow \qquad C_{\dot{\alpha}(2s)} \tag{C.58f}$$

where $(N - \bar{N})\mathcal{F}(y, \bar{y}) = (N - \bar{N})\mathcal{F}'(y, \bar{y}) = 0$. It is therefore natural to identify $\mathcal{F}$ and $\mathcal{F}'$ with the traceless and trace components of the Fronsdal tensor which correspond to $F_{\alpha(s)\dot{\alpha}(s)}$ and $F'_{\alpha(s-2)\dot{\alpha}(s-2)}$ in spinorial notation. Similarly, we can expect that $\mathcal{C}(y)$ and $\bar{\mathcal{C}}(\bar{y})$ encode the spin-s Weyl tensor which in spinorial notation is given by $C_{\alpha(2s)}$ and $C_{\dot{\alpha}(2s)}$.

Therefore, we expect that unfolded equation (C.46) imposes the Fronsdal equations and identifies the linearized Weyl tensors with the appropriate components of the zero-form (up to an overall normalization)[10]. We will show this explicitly in the next section.

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9 The case spin one is degenerate: the one-form ω_0 only consists of the Y-independent component which encodes the dynamical spin-1 field, i.e. $\omega_0 = A(x)$.

10 The case of spin one is again degenerate: the unfolded equation splits into a single component

$$\mathrm{d}A = E^{\alpha\alpha} F_{\alpha\alpha} - E^{\dot{\alpha}\dot{\alpha}} F_{\dot{\alpha}\dot{\alpha}} \,,$$

C.3.2.3 *Weyl Tensors, Fronsdal Fields and Equations*

FRONSDAL FIELDS: In the last section, we discussed that at fixed spin $s > 1$ the dynamical fields are contained in the one-form as

$$\omega_0(y, \bar{y}) = \bar{e}^{\alpha\dot{\alpha}} \left(\partial_\alpha \partial_{\dot{\alpha}} \, \phi(y, \bar{y}) + y_\alpha \bar{y}_{\dot{\alpha}} \, \phi'(y, \bar{y})\right) . \tag{C.59}$$

Because we are considering fixed spin s, it follows that $N\phi = \bar{N}\phi = s\,\phi$ and $N\phi' = \bar{N}\phi' = (s-2)\,\phi'$. As we also discussed, we expect that ϕ and ϕ' encode the traceless and trace component of the Fronsdal field. We now show that this identification indeed leads to the expected gauge transformations. This can be seen by considering the gauge transformation of the one-form

$$\delta\omega_0(y, \bar{y}) = -\tfrac{1}{2}\bar{e}^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \xi_0 + Q_- \xi_{+1} + Q_+ \xi_{-1} \,. \tag{C.60}$$

The last two components do not contribute to the gauge transformations of the dynamical fields since $Q_\pm \xi_{\mp 1}|^{\partial\bar{\partial}} = Q_\pm \xi_{\mp 1}|^{y\bar{y}} = 0$. Using the identity (C.40) twice, we obtain

$$\delta\omega_0 = -\tfrac{1}{2}\bar{e}^{\alpha\dot{\alpha}} \left(\partial_\alpha \partial_{\dot{\alpha}} \tfrac{1}{N\bar{N}} y^\beta \bar{y}^{\dot{\beta}} \nabla_{\beta\dot{\beta}} \xi_0 + y_\alpha y_{\dot{\alpha}} \tfrac{1}{(N+2)(\bar{N}+2)} \nabla^{\beta\dot{\beta}} \partial_\beta \partial_{\dot{\beta}} \xi_0 + \ldots \right) ,$$

where we have only written out components which will contribute to the gauge transformations of the dynamical fields. In components this implies that

$$\delta\phi_{\alpha(s)\dot{\alpha}(s)} \sim \nabla_{\alpha\dot{\alpha}} \xi_{\alpha(s-1)\dot{\alpha}(s-1)} \,, \tag{C.61}$$

$$\delta\phi'_{\alpha(s-2)\dot{\alpha}(s-2)} \sim \nabla^{\beta\dot{\beta}} \xi_{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-2)} \,, \tag{C.62}$$

which shows that these components indeed coincide with the traceless and trace component of the Fronsdal field up to spin-dependent normalization factors.

WEYL TENSORS: We will now show that the on-mass-shell theorem (6.59) indeed determines the Weyl tensor components $C^{\alpha(2s)}$, $C^{\dot{\alpha}(2s)}$ of the zero-form in terms of the Fronsdal field. At fixed spin s, the on-mass-shell theorem equation for the one-form decomposes as

$$\nabla\omega_k + Q_+\omega_{k-1} + Q_-\omega_{k+1} = \mathcal{V}(\Omega, \Omega, C^{(2)})\delta_{k,\pm(s-1)} \tag{C.63}$$

for $-(s-1) \leq k \leq s-1$ and

$$\mathcal{V}(\Omega, \Omega, C^{(2)}) = \frac{1}{2} e^{i\theta} E^{\alpha\alpha} \, \partial_\gamma \partial_\gamma \, C_{+s} - \frac{1}{2} e^{-i\theta} E^{\dot{\gamma}\dot{\gamma}} \, \partial_{\dot{\gamma}} \partial_{\dot{\gamma}} \, C_{-s} \,. \tag{C.64}$$

We will now solve this equation for C_s in terms of ϕ (and in principle also ϕ' which however will not contribute). For $s = 1$ the equation of

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where we have defined $A(x) = \omega_0$ and $F_{\alpha\alpha} = -\frac{1}{2}\, e^{i\theta} C_{\dot{\alpha}\dot{\alpha}}$. Therefore, there is no need for a cohomological analysis and the equation simply defines the field-strength tensor F_{nm} in terms of the spin-one field.

motion simply defines the field-strength tensor in terms of the spin-1 field - see (6.58). For $s > 1$ and $k = 0$ the equation of motion for ω_k are given by

$$\nabla\omega_0 + Q_+\omega_{-1} + Q_-\omega_{+1} = 0\,. \tag{C.65}$$

By the analysis of the σ_--cohomology this is solved by $\omega_{-1} = -Q_+^{\#}\omega_{-1}$ and $\omega_{+1} = -Q_-^{\#}\omega_0$, where the operators $Q_\pm^{\#}$ were defined in (6.112). Plugging this in the equation of motion with $k = 1$ we obtain

$$-\nabla(Q_-^{\#}\omega_0) + Q_+\omega_0 + Q_-\omega_{+2} = 0\,. \tag{C.66}$$

which is solved by $\omega_{+2} = (-Q_-^{\#}\nabla)^2\omega_0$ because of $Q_-^{\#}Q_+\bullet = 0$ (see (6.112)). This process continues and gives

$$\omega_{+k} = (-Q_-^{\#}\nabla)^k\omega_0 \tag{C.67}$$

until $k = s-1$ for which the equation of motion becomes

$$(-\nabla Q_-^{\#})^{s-1}\nabla\omega_0 + Q_+\omega_{s-2} + Q_-\omega_s = \mathcal{V}(\Omega,\Omega,C^{(2)})_{s-1}\,. \tag{C.68}$$

We now restrict to its $J^{\partial\partial}$-component of the decomposition (C.57) using the notation $J|^{\partial\partial} = J^{\partial\partial}$. Since the last two summands on the left hand side do not contribute to this component, we obtain

$$(-\nabla Q_-^{\#})^{s-1}\nabla\omega_0\Big|^{\partial\partial} = -\frac{1}{2}e^{i\theta}C_s\,. \tag{C.69}$$

In order to simplify this expression we will first focus on the case of $s = 2$. We note that

$$-\nabla Q_-^{\#}J\Big|^{\partial\partial} = -\nabla Q_-^{\#}E^{\alpha\alpha}\partial_\alpha\partial_\alpha J^{\partial\partial}\Big|^{\partial\partial}\,, \tag{C.70}$$

for a two-form J. This is because

$$\begin{aligned}\nabla Q_-^{\#}J\Big|^{\partial\partial} &= -\tfrac{1}{2}\bar{e}^{\alpha\dot\alpha}\wedge\bar{e}^{\beta\dot\beta}\nabla_{\alpha\dot\alpha}\left(\partial_\beta\partial_{\dot\beta}\frac{2}{\bar{N}}J^{\partial\partial} + y_\beta\bar{y}_{\dot\beta}\cdots + y_\beta\partial_{\dot\beta}\ldots\right)\Big|^{\partial\partial}\\ &= -\tfrac{1}{4}E^{\alpha\alpha}\left(\partial_\alpha\nabla_\alpha{}^{\dot\gamma}\partial_{\dot\gamma}\frac{2}{\bar{N}}J^{\partial\partial} + y_\alpha\nabla_\alpha{}^{\dot\gamma}\bar{y}_{\dot\gamma}\cdots + y_\alpha\nabla_\alpha{}^{\dot\gamma}\partial_{\dot\gamma}\ldots\right)\Big|^{\partial\partial}\end{aligned}$$

where in the first line we have used the definition of $Q_-^{\#}$ in (6.112) and only made terms explicit which will be relevant in the end. Using the decomposition identity (C.40), we see that the last two terms will drop out and the first gives

$$\nabla Q_-^{\#}J\Big|^{\partial\partial} = \frac{1}{2N(\bar{N}+1)}(y\nabla\bar\partial)J^{\partial\partial}\,, \tag{C.71}$$

where we have defined $y\nabla\bar\partial = y^\alpha\nabla_{\alpha\dot\alpha}\bar\partial^{\dot\alpha}$. Using the analogous decomposition identity (6.123) for two-forms it is easy to check that

$$\nabla\omega_0|^{\partial\partial} = -\frac{1}{4N}(y\nabla\bar\partial)\phi\,, \tag{C.72}$$

where we have used the embedding of the dynamical field (C.59) in the one-form ω. Note that this result in particular implies that only the traceless components ϕ of the Fronsdal fields contribute. Using these results (C.69) becomes

$$-\frac{1}{2}e^{i\theta}C_s = -\frac{1}{2N(\bar{N}+1)}(y\nabla\bar{\partial})^{s-1}\frac{1}{4N}(y\nabla\bar{\partial})\phi \tag{C.73}$$
$$= \mathcal{N}_s\,(y\nabla\bar{\partial})^s\,\phi\,, \tag{C.74}$$

where $\mathcal{N}_s$ is a spin-dependent normalization constant. In components this implies

$$C_{\alpha(2s)} \sim (\nabla_\alpha{}^{\dot{\alpha}})^s\phi_{\alpha(s)\dot{\alpha}(s)}\,. \tag{C.75}$$

Analogously, one can derive

$$C_{\dot{\alpha}(2s)} \sim (\nabla^\alpha{}_{\dot{\alpha}})^s\phi_{\alpha(s)\dot{\alpha}(s)}\,. \tag{C.76}$$

Using the map from spinorial to Lorentz indices discussed in Section 6.1 these equations are equivalent to

$$C^{a(s),b(s)} \sim [\text{anti-sym. in } b \text{ and } a]\ \nabla^{b_1}...\nabla^{b_s}\phi^{a_1...a_s} - \text{traces}\,, \tag{C.77}$$

which is indeed what one expects for the linearized spin-s Weyl tensors.

FRONSDAL EQUATIONS: Fronsdal equations can be obtained along very similar lines. The relevant components of the unfolded equations (C.46) are given by[11]

$$R_{+1} := \nabla\omega_{+1} + Q_+e + Q_-\omega_{+2} = 0\,, \tag{C.78a}$$
$$R_0 := \nabla e + Q_+\omega_{-1} + Q_-\omega_{+1} = 0\,, \tag{C.78b}$$
$$R_{-1} := \nabla\omega_{-1} + Q_+\omega_{-2} + Q_-e = 0\,, \tag{C.78c}$$

where we have defined $e := \omega_0$. From the analysis of the σ_--cohomology, we expect that the Fronsdal tensor is embedded in the curvatures as follows

$$R_{+1}\Big|_F := E^{\dot{\alpha}\dot{\alpha}}\partial_{\dot{\alpha}}\partial_{\dot{\alpha}}F + E^{\alpha\alpha}y_\alpha y_\alpha F'\,,$$
$$R_{-1}\Big|_F := E^{\alpha\alpha}\partial_\alpha\partial_\alpha F + E^{\dot{\alpha}\dot{\alpha}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\alpha}}F'\,,$$

From the previous arguments, it is clear that the generalized torsion constraint is solved by

$$\omega_{-1} = -Q_+^{\#}(\nabla e)\,, \qquad \omega_{+1} = -Q_-^{\#}(\nabla e)\,. \tag{C.80}$$

Plugging the last expression in R_{+1} we obtain

$$R_{+1} = -\nabla Q_-^{\#}\nabla e + Q_+e + Q_-\omega_{+2} = 0 \tag{C.81}$$

11 Here we assume $s > 2$. For $s = 2$ there would be a source term on the right hand side of $R_{\pm1}$. However, this source term will not affect the argument in any way.

We now use that $Q_- \bullet |^{\bar\partial\bar\partial} = 0$ and $Q_- \bullet |^{yy} = 0$ to obtain

$$F = \left[-\nabla Q_-^{\#} \nabla e + Q_+ e\right]^{\bar\partial\bar\partial} , \tag{C.82}$$

$$F' = \left[-\nabla Q_-^{\#} \nabla e + Q_+ e\right]^{yy} . \tag{C.83}$$

These expressions are in principle straightforward to work out using the techniques discussed for the case of the Weyl tensors and indeed reproduce the traceless and trace component of the Fronsdal tensor (up to spin dependent normalization constants). However, this calculation is a bit involved and we will not reproduce it here. The relevant normalization constants (using slightly different conventions!) can be found in Appendix C.2 of our publication [27].

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Hiermit erkläre ich meine Dissertation selbstständig ohne fremde Hilfe verfasst zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Pan Kessel

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